Forward scattering series and seismic events: high frequency approximations, critical and postcritical reflections

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Abstract

In this paper we progress the analysis of the forward scattering series and seismic events for 1D normal incidence seismic data introduced by Matson (1996) and later extended to prestack data by Matson (1997). We show that the exchange of certain integrals in the prestack calculation of the forward scattering series terms in the later yield the same result as a high frequency approximation of the integrals, without the interchange. The reasonableness of this outcome is described from both a mathematical and a physical point of view. The convergence of the forward scattering series at the critical angle is also proved and an explanation is proposed for the divergence of the series for postcritical incident planewaves.

1 Introduction

Inverse scattering series is using the deghosted and demultipled recorded data to locate reflectors and invert for parameters that change at those locations. The present location and inversion algorithms, although successful, are written for smooth, horizontal interfaces and process either 1D data or 2D precritical data. If interfaces are considered to have a higher degree of realism (dipping, highly corrugated, diffractive etc.) then the data contains more and different information than expected by those algorithms (postcritical reflections, headwaves) even for simple incident planewaves. For both fundamental and practical advances, it is important to establish the attitude taken towards these returning signals (portions of the recorded scattered wavefield) as noise or as information bearing agents. The general objective of this particular research is to establish the scattering theory description and processing of more complicated waves. While analogies between forward and inverse processes are not maps, they never-the-less provide useful hints or at least point where certain activity resides in the inverse series. The forward series does not hint at whether events will be signal or noise in the inverse series, only suggests where one might look for that answer. Take multiples for example: it turns out that the inverse scattering subseries made of terms that mimic the diagrams for multiples in the forward series is responsible for attenuating/removing them from the data. Precritical data has been studied by Matson (1997) who showed that the expected (from wave-theory) precritically reflected wavefield is constructed by the convergent forward scattering series in a 2D experiment. This study brings new understanding about the physical interpretation of Matson results; it also shows that the same forward series converges for critical angles as well. An explanation of the postcritical divergence is proposed.

2 The forward series and seismic reflection data

Matson (1996) and thesis (1997) describe the propagation of a wavefield in a given 1D or 2D medium respectively, using the forward scattering series. The actual medium is viewed as a perturbation of a reference medium; the propagation of the wavefield in the actual medium is given by the forward scattering (Born) series, a series of propagations in the reference medium separated by different orders of scattering interactions with a point scatterer perturbation

$$\begin{aligned} P(\vec{r}|\vec{r}_s;\omega) &= P_0(\vec{r}|\vec{r}_s;\omega) \\ &+ \int_{-\infty}^{\infty} P_0(\vec{r}|\vec{r}';\omega)V(\vec{r}')P_0(\vec{r}'|\vec{r}_s;\omega)d\vec{r}' \\ &+ \int_{-\infty}^{\infty} P_0(\vec{r}|\vec{r}';\omega)V(\vec{r}')\left(\int_{-\infty}^{\infty} P_0(\vec{r}'|\vec{r}'';\omega)V(\vec{r}'')P_0(\vec{r}''|\vec{r}_s,\omega)d\vec{r}''\right)d\vec{r}' \\ &+ \ldots. \end{aligned}$$

where all the position vectors are fully 3D and $\vec{r}_s$ represents the position of the source, $\vec{r}$ the position of the receiver and $\vec{r}'$ indicates the position of a point scatterer. For a two dimensional model, the propagations in the reference medium are given by the 2D Green's function

$$G_0(x,z|x_s,z_s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik_s(x-x_s)}e^{i\nu_{0s}|z_g-z_s|}}{2i\nu_{0s}}dk_s$$

where k_s and ν_{0s} are the horizontal and the vertical wavenumber of the reference medium respectively ($\nu_{0s}^2 + k_s^2 = \omega^2/c_0^2$). Rewriting G_0 as

$$G_0(x,z|x_s,z_s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ik_s x_s}}{2i\nu_{0s}}\phi_0(x_g,z_g|k_s,z_s;\omega)dk_s$$

with $\phi_0(x_g, z_g|k_s, z_s; \omega) = e^{i(k_s x_g + \nu_{0s}|z_g - z_s|)}$, it is apparent that G_0 represents a superposition of weighted plane waves. This motivates the use of a plane-wave as the incident wave with the remark that one can construct solutions for point sources from plane-wave solutions by performing the above mentioned weighted integration. The model used is a half-space earth with no lateral variance, with an interface at z_1 ; the scattering perturbation for this model is

$$V(z') = k_0^2 a_1 H(z' - z_1)$$

where $a_1 = 1 - c_0^2/c_1^2$, c_1 is the velocity in the second medium, c_0 the velocity in the reference medium with H being the Heaviside function. For simplicity consider the source location to be (0, 0); the Born series takes the form

$$\begin{aligned} P(\vec{r}|\vec{r}_s; \omega) \;=\; & e^{i(kx_g + \nu_0 z_g)} \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_0|z_g - z'|}}{2i\nu_0} dk_g k_0^2 a_1 H(z' - z_1) P_0(x', z'; \omega) dx' dz' \\ & + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_0|z_g - z'|}}{2i\nu_0} dk_g k_0^2 a_1 H(z' - z_1) P_1(x', z' > z_1|k, \omega) dx' dz' \\ & + \ldots . \end{aligned}$$

Note that the incoming wave hits all the scatterers at once; each scatterer then emits a spherical wave which can propagate to the receiver or to another point scatterer. Each term in the forward series represents the response at the receiver after a certain number of interactions: the zeroth term represents the direct arrival, the first term represents the wavefield after one interaction with a point scatterer and so on. To construct even the simplest event, one needs an infinite number of terms in the forward series. To obtain the total wavefield at the receiver we have to solve the integrals in the previous expression. Following Matson (1997) we solve for the first term in the series

$$P_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_0|z_g - z'|}}{2i\nu_0} dk_g k_0^2 a_1 H(z' - z_1) e^{i(kx' + \nu_0 z')} dx' dz'$$

Begin by switching the order of integration so that the integration with respect to dx' is performed first. Hence

$$P_1 = \frac{1}{2\pi}\int_{z_1}^{\infty}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{i(k - k_g)x'} dx' \right) e^{ikgx_g} e^{i\nu_{0g}|z_g - z'|} e^{i\nu_0 z'} \frac{k_0^2 a_1}{2i\nu_{0g}} dk_g dz'$$

Using

$$\int_{-\infty}^{\infty} e^{i(k-k_g)x'}dx' = 2\pi\delta(k_g - k)$$

P_1 becomes

$$P_1 = \int_{z_1}^{\infty}\int_{-\infty}^{\infty} \delta(k_g - k)e^{ikgx_g}e^{i\nu_{0g}|z_g-z'|}e^{i\nu_0 z'}\frac{k_0^2 a_1}{2i\nu_{0g}}dk_g dz'.$$

Using the properties of the delta function we see that the inside integral switches $k_g \to k$ and hence $\nu_{0g} \to \nu_0$ so the expression becomes

$$P_1 = \frac{k_0^2 a_1}{2i\nu_0}e^{ikx_g}\int_{z_1}^{\infty} e^{i\nu_0|z_g-z'|}e^{i\nu_0 z'}dz'.$$

There are two cases to be considered at this point: $z_g < z_1$ for the reflected P_1 and $z_g > z_1$ for the transmitted part. The first enters into the series for the total reflected field while the second is used either into the series for transmitted wavefield or for the calculation of P_2 (reflected or transmitted).
We have

$$P_1(x_g, z_g < z_1|k;\omega) = \frac{k_0^2 a_1}{2i\nu_0}e^{ikx_g}e^{-i\nu_0 z_g}\int_{z_1}^{\infty} e^{i\nu_0 2z'}dz'.$$

Integrating and considering that, due to some small dissipation, the wave at infinity vanishes, we obtain

$$P_1(x_g, z_g < z_1|k;\omega) = \frac{k_0^2 a_1}{4\nu_0^2}e^{ikx_g}e^{i\nu_0(2z_1-z_g)}.$$

Same integration procedure is used for the calculation of P_2 , P_3 etc. The calculated terms

$$P(x_g, z_g < z_1|k;\omega) = e^{ikx_g}e^{i\nu_0(2z_1-z_g)}\left[\frac{1}{4}\frac{k_0^2 a_1}{\nu_0^2} + \frac{1}{8}\left(\frac{k_0^2 a_1}{\nu_0^2}\right)^2 + \frac{5}{64}\left(\frac{k_0^2 a_1}{\nu_0^2}\right)^3 + \frac{7}{128}\left(\frac{k_0^2 a_1}{\nu_0^2}\right)^4 + ...\right]$$

indicate certain regularity: the series is recognized to be the Taylor series for $\sqrt{1 - \frac{k_0^2 a_1}{\nu_0^2}}$ about $\frac{k_0^2 a_1}{\nu_0^2} = 0$ after some algebraic operations performed on it. The ratio test indicates that the series converges for $\left|\frac{k_0^2 a_1}{\nu_0^2}\right| < 1$. By writing $\nu_0 = k_0 \cos\theta$, with θ being the incidence angle of the plane wave, this condition becomes

$$\sin\theta < \frac{c_0}{c_1} < \left(1 + \cos^2\theta\right)^{1/2} \tag{1}$$

This last relation can be viewed in two ways:

1. First, for a fixed incidence angle θ, this is a restriction on the velocity contrast between the reference and the actual medium. For $\theta = 0$ (normal incidence) the left inequality is satisfied for any 2 velocities; the right hand inequality becomes $c_0 < \sqrt{2}c_1$ (Matson 1996).

2. Second, for a fixed velocity model, the restriction is on the incident angle. Note that given any two velocities c_0 and c_1, one of the two inequalities is automatically satisfied. For $c_0 > c_1$, the condition reads $\frac{c_0}{c_1} < (1 + \cos^2\theta)^{1/2}$ or $\sin^2\theta < 1 + a_1$ with $a_1 < 0$.

For $c_0 < c_1$, the condition becomes $\sin\theta < \frac{c_0}{c_1}$ or $\theta < \theta_c$ where θ_c is the critical angle $\theta_c = \sin^{-1}(c_0/c_1)$. When the series converges, the limit is

$$2\frac{\nu_0^2}{k_0^2 a_1}\left[1 - \sqrt{1 - \frac{k_0^2 a_1}{\nu_0^2}}\right] - 1 = \frac{\nu_0 - \nu_1}{\nu_0 + \nu_1}$$

so the final expression for the total field is

$$P = \frac{\nu_0 - \nu_1}{\nu_0 + \nu_1} e^{ikx_g} e^{i\nu_0(2z_1 - z_g)}$$

which is the expected result from Wave Theory.

3 Comments about the calculation of terms in the forward scattering series

The calculation of

$$P_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_0|z_g - z'|}}{2i\nu_0} dk_g\right) k_0^2 a_1 H(z' - z_1) e^{i(kx' + \nu_0 z')} dx' dz' \tag{2}$$

contains a reordering of integrals: in the original expression the dk_g integral should be solved first since it is the inner most one then the dx' integral and finally the dz' one.

The calculations are greatly simplified if the dx' integration is performed first, then the dk_g and finally the dz' integration. However this kind of operation has to be performed with great care since it might change the result obtained from solving the integrals in the original order. In this section we show that the interchange of integrals is only valid in the high frequency approximation. To get a feeling why this statement is true even before proving it mathematically, study the equation (2) in detail. Recall that Fubini theorem gives sufficient conditions for interchanging integrals. It states that when a function f is integrable (continuity or boundedness imply integrability) on $R^n = R^k \times R^m$ then the iterated integrals of f over R^k and R^m exist and

$$\int_{R^n} f = \int_{R^k}\int_{R^m} f(x,y)dydx = \int_{R^m}\int_{R^k} f(x,y)dxdy.$$

Note that the inner most integrand in (2) is neither continuous nor bounded due to $\nu_0 = \sqrt{\frac{\omega^2}{c_0^2} - k_g^2}$ in the denominator. When k_g runs from $-\infty$ to ∞, the integrand blows up at $\pm\frac{\omega}{c_0}$. However, in the high frequency approximation, we have

$$\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g-x')}e^{i\nu_0|z_g-z'|}}{2i\nu_0} dk_g \sim \int_{-\omega/c_0}^{\omega/c_0} \frac{e^{ik_g(x_g-x')}e^{i\nu_0|z_g-z'|}}{2i\nu_0} dk_g \qquad (3)$$

and the integrand is now continuous hence integrable everywhere. The Fubini theorem applies and the interchange of integrals, and hence Matson's derivation, is valid. Next, we recalculate P_1 using saddle point approximations for the two integrals involved without switching the order of integration and show that the result is the one obtained by Matson. Saddle point or stationary phase approximation gives the leading asymptotic behavior of generalized Fourier integrals, i.e. of the form $\int_{-\infty}^{\infty} F(p)e^{\omega f(p)}dp$, having stationary points, i.e. points p_s such that $f'(p_s) = 0$. The idea of the method is to use the analyticity of the integrand to justify deforming the path of integration to a new path on which $f(p)$ has a constant imaginary path. How the contour is deformed depends on the singularities and branch cuts of the integrand. Once this has been done the integral may be found asymptotically ($\omega \to \infty$) to be

$$\int_{-\infty}^{\infty} F(p)e^{\omega f(p)}dp \sim \left|\frac{2\pi}{\omega f''(p_s)}\right|^{1/2} F(p_s)e^{isign(f''(p_s))\frac{\pi}{4}} \exp\left[\omega f(p_s)\right]. \qquad (4)$$

To calculate P_1 start by rewriting

$$G_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g-x')}e^{i\nu_0|z_g-z'|}}{2i\nu_0} dk_g$$

as

$$G_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) e^{\omega f(p)} dp$$

where

$$F(p) = \frac{1}{2i\sqrt{1/c_0^2 - p^2}}$$

and

$$f(p) = i \left[p(x_g - x') + \sqrt{\frac{1}{c_0^2} - p^2}\, |z_g - z'| \right].$$

Note that, due to the square root, $F(p)$ defines two branch cuts in the complex p plane; the branch cuts are hyperbolas in the first and third quadrant and are running very close to the coordinate axis (for a full discussion of the branch cuts of F see Aki and Richards (1980) Box 6.2). By definition, branch cuts are lines of discontinuities for $F(p)$ and here are given by $\mathrm{Im}\sqrt{1/c_0^2 - p^2} = 0$. This means that when the new integration path (see Figure 6.6 in Aki and Richards) intersects these branch cuts, $F(p)$ is discontinuous hence not analytic. This apparent problem can be avoided if we relax the condition $\mathrm{Im}\sqrt{1/c_0^2 - p^2} \geq 0$ along the integration path. Instead we allow $\mathrm{Im}\sqrt{1/c_0^2 - p^2}$ to change sign at each branch cut intersection which, for the integration path, is equivalent to a transition to a different Riemann sheet. The integrand looses physical interpretation while on another Riemann sheet but gains analyticity. However, the two intersections with the branch cut insure two sign changes and the emergence of the integrand with the correct sign at the saddle point (eventually the integrand is going to be expanded in a Taylor series at that point and the rest of the path is going to be discarded). To calculate the location of the saddle point equate the derivative of f with zero; we find

$$p_s = \frac{x_g - x'}{c_0 d'}$$

with $d' = \sqrt{(z_g - z')^2 + (x_g - x')^2}$. Calculate

$$f(p_s) = i\frac{d'}{c_0}$$

$$f''(p_s) = -\frac{i c_0 d'^3}{|z_g - z'|^2}$$

$$F(p_s) = \frac{c_0 d'}{2i\,|z_g - z'|}$$

and plug them into the above formula (4) to obtain

$$G_0 \sim \frac{1}{2\pi i}\left(\frac{\pi c_0}{2\omega d'}\right)^{1/2} e^{ik_0 d'}. \tag{5}$$

This last expression shows that, in the high frequency approximation, the main contribution from each point scatterer comes along the line connecting the scatterer with the receiver. The coefficient accounts for the dismissed directions of propagation. In time domain the G_0 from equation (5) is

$$G_0 = \frac{1}{i}\sqrt{\frac{c_0}{2id'}}\frac{1}{\sqrt{t - \frac{d'}{c_0}}} H\left(t - \frac{d'}{c_0}\right). \tag{6}$$

With this approximation, the expression for P_1 becomes

$$P_1 = \frac{1}{2\pi i}\int_{z_1}^{\infty}\int_{-\infty}^{\infty} e^{ik_0 d'}\left(\frac{\pi c_0}{2\omega d'}\right)^{1/2} k_0^2 a_1 e^{i(kx' + \nu_0 z')} dx' dz' \tag{7}$$

or

$$P_1 = \frac{k_0^{3/2} a_1}{2\pi i}\sqrt{\frac{\pi}{2}}\int_{z_1}^{\infty} e^{i\nu_0 z'}\int_{-\infty}^{\infty}\frac{e^{i\omega\left(\frac{d'}{c_0} + \frac{k}{\omega}x'\right)}}{\sqrt{d'}} dx' dz'. \tag{8}$$

Again, the inner most integral has the form

$$I = \int_{-\infty}^{\infty} F(x') e^{\omega f(x')} dx'$$

with $F(x') = \frac{1}{\sqrt{d'}}$ and $f(x') = i\left(\frac{d'}{c} + \frac{k}{\omega}x'\right)$. Note that the integrand has no branch cuts this time since $d' = \sqrt{(z_g - z')^2 + (x_g - x')^2}$ is always positive; the saddle point is x'_s such that

$$x_g - x'_s = |z_g - z'|\frac{k'}{\nu_0}$$

and so we have

$$f(x'_s) = \left(\frac{\nu_0}{\omega}|z_g - z'| + \frac{k}{\omega}x_g\right),$$

$$f''(x'_s) = \frac{ic_0^2\nu_0^3}{\omega^3\,|z_g - z'|}$$

and

$$F(x'_s) = \frac{1}{\sqrt{|z_g - z'|}}\sqrt{\frac{c_0\nu_0}{\omega}}.$$

Using the same high frequency approximation (4) we find

$$\int_{-\infty}^{\infty} \frac{e^{i\omega\left(\frac{d'}{c}+\frac{k}{\omega}x'\right)}}{\sqrt{d'}}dx' \sim \frac{1}{\nu_0}\sqrt{\frac{2\pi\omega}{c_0}}e^{i(\nu_0|z_g-z'|+kx_g)}. \tag{9}$$

Plugging this into the expression (8) for P_1 we obtain

$$P_1 = \frac{k_0^2 a_1}{2i\nu_0}e^{ikx_g}\int_{z_1}^{\infty} e^{i\nu_0|z_g-z'|}e^{i\nu_0 z'}dz'$$

which is the same result as obtained before by switching the order of integration. The rest of the terms in the series for P can be similarly shown to resemble the expressions given by Matson (1997).

4 Physical interpretation of the approximations

The two high frequency approximations made in the previous derivation have an easy to understand physical interpretation. The approximation of the first integral represents the most important contribution coming to the receiver from each point scatterer (see Figure 1).

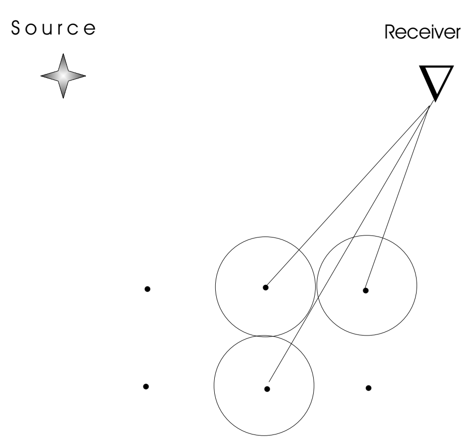

Figure 1: The physical interpretation of the approximation of G_0 - the inner most integral in the expression of P_1.

The result,

$$G_0 \sim \frac{1}{2\pi i}\left(\frac{\pi c_0}{2\omega d'}\right)^{1/2} e^{ik_0 d'},$$

represents the ray from the scatterer to the receiver multiplied by a coefficient which accounts for the dismissal of all the other directions of propagations.

The approximation of the second integral looks at the totality of the rays arriving at the receiver. Again the most important contribution is picked out to be the one coming from the rays that make an angle equal to the incident's plane wave angle with the vertical (see Figure 2); this can also be seen from the expression of the saddle point

$$x_g - x_s' = |z_g - z'|\,\frac{k'}{\nu_0}. \tag{10}$$

The last integral in the expression of P_1 is a one dimensional integral along the bold line

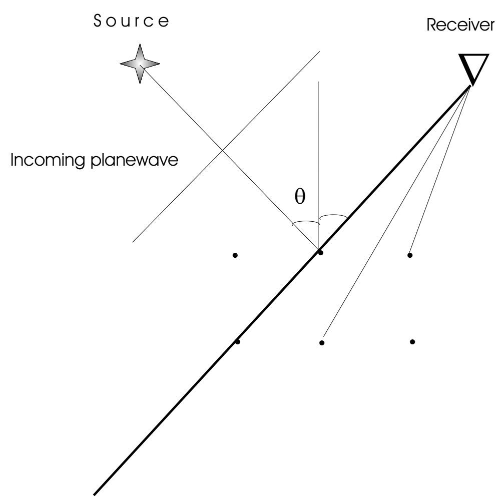

Figure 2: The physical interpretation of the approximation of the second integral in the calculation of P_1.

showed in Figure 2. Even though the parameter of integration is z', there is a certain relation between z' and x', given by (10), so that the direction of integration is tilted at angle equal to the incident angle and not vertical.

5 Convergence at the critical angle

The forward scattering series for the model discussed in this paper is (see Matson (1997))

$$P(x_g, z_g < z_1|k;\omega) = e^{ikx_g} e^{i\nu_0(2z_1 - z_g)} \left[\frac{1}{4}\frac{k_0^2 a_1}{\nu_0^2} + \frac{1}{8}\left(\frac{k_0^2 a_1}{\nu_0^2}\right)^2 + \frac{5}{64}\left(\frac{k_0^2 a_1}{\nu_0^2}\right)^3 + \frac{7}{128}\left(\frac{k_0^2 a_1}{\nu_0^2}\right)^4 + ...\right]. \tag{11}$$

The ratio test shows convergence for $\left|\frac{k_0^2 a_1}{\nu_0^2}\right| < 1$, divergence for $\left|\frac{k_0^2 a_1}{\nu_0^2}\right| > 1$ and it is inconclusive for $\left|\frac{k_0^2 a_1}{\nu_0^2}\right| = 1$. When $c_0 < c_1$ this last condition is equivalent to $\frac{k_0^2 a_1}{\nu_0^2} = 1$ which in turns is equivalent to $\theta = \theta_c$, i.e. the incident angle is the critical angle. In other words, the forward series is convergent for precritical incidence and divergent for postcritical incidence; no information is found about the critical incidence. For a critical incident planewave, the series becomes

$$P(x_g, z_g < z_1|k;\omega) = e^{ikx_g} e^{i\nu_0(2z_1 - z_g)} \left[\frac{1}{4} + \frac{1}{8} + \frac{5}{64} + \frac{7}{128} + ...\right]. \tag{12}$$

Rewrite

$$R = \frac{1}{4} + \frac{1}{8} + \frac{5}{64} + \frac{7}{128} + ... = \sum_{n=2}^{\infty} \frac{1}{n!}\frac{1\cdot 3\cdot 5....(2n-3)}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)}{(n+1)!\Gamma(1/2)}.$$

Note that the series has the form $\sum_{n=2}^{\infty} a_n$ with $a_n = \frac{1}{n!}\frac{1\cdot 3\cdot 5....(2n-3)}{2^{n-1}}$ and so

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n-1} - 1\right) = \frac{3}{2} > 1$$

hence Raabe's Convergence Test shows convergence (for a full discussion of this convergence test see Bromwich (1965)). The conclusion is that the forward scattering series for this model converges at the critical angle as well. Note that the sum of the series, which corresponds to the reflection coefficient, is $R = 1$.

6 Postcritical divergence

For a $c_0 < c_1$ model, the forward series converges for precritical and critical incidence and diverges for postcritical incidence. From wave theory, the reflection coefficient R which should be constructed by the forward scattering series is:

- $R = \frac{\nu_0 - \nu_1}{\nu_0 + \nu_1} < 1$ for precritical incidence. In this case both ν_0 and ν_1 are real.
- $R = 1$ for critical incidence. In this case $\nu_1 = 0$.
- $R = \frac{\nu_0 - \nu_1}{\nu_0 + \nu_1}$ for postcritical incidence. In this case ν_1 is purely imaginary and hence R is complex. However $|R| = 1$ and the complexity of R is attributed to a phase-shift of the emerging wave after hitting the interface due to the evanescent waves created in the second medium.

The term $\frac{a_1 k_0^2}{\nu_0^2} = 1 - \frac{\nu_1^2}{\nu_0^2}$ is grater than one exactly when ν_1 becomes imaginary. In fact if for this case one writes $R = e^{i\epsilon}$, where ϵ is the phase-shift of the wavefield, then $\frac{a_1 k_0^2}{\nu_0^2} = 1 + \tan^2 \frac{\epsilon}{2}$ enforcing the earlier statement that the divergence is due to the phase-shift of the reflected wave. In other words, it is the impossibility of constructing a complex number ν_1 as a series of real numbers (powers of ν_0) which leads to the divergence of the series. The graph of ν_1 as a function of $\frac{a_1 k_0^2}{\nu_0^2}$ is shown in Figure 3. For $c_0 < c_1$ we have that $a_1 > 0$ so we are looking at

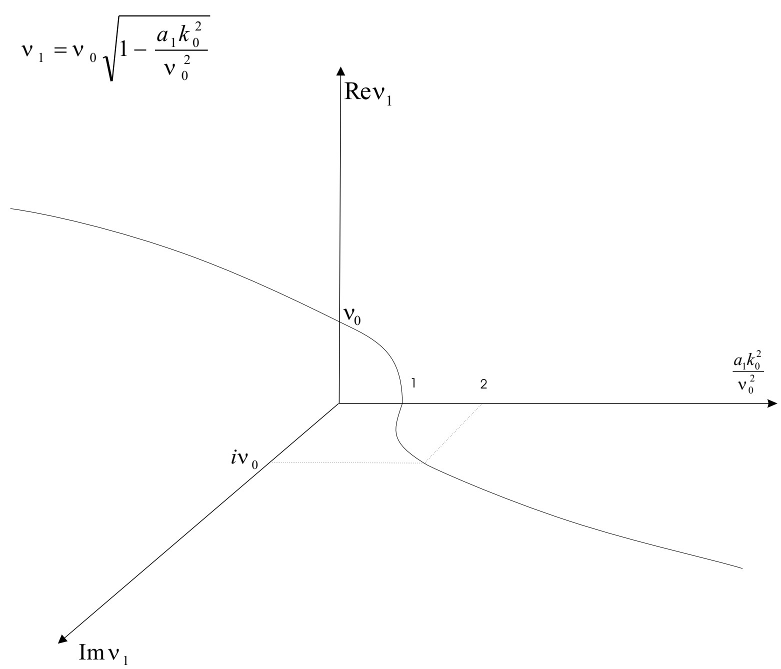

Figure 3: The graph of ν_1 as a function of $\frac{a_1 k_0^2}{\nu_0^2}$.

the positive x-axis of the graph; if the velocity model is fixed, $a_1 k_0^2$ is a constant. The vertical wavenumber of the propagating wave in the actual medium, ν_1, is equal to ν_0 when $\frac{a_1 k_0^2}{\nu_0^2} = 0$, i.e. at normal incidence. When $\frac{a_1 k_0^2}{\nu_0^2} = 1$ (at critical incidence), ν_1 is zero showing that there is no propagation into the second medium. When $\frac{a_1 k_0^2}{\nu_0^2} > 1$ (post-critical incidence), ν_1 is complex and it becomes unrecoverable by a Taylor series written at $\frac{a_1 k_0^2}{\nu_0^2} = 0$; the series is now divergent. For $c_0 > c_1$ it seems like this problem does not exist. In this case there is no critical angle and so the vertical wavenumber ν_1 never becomes complex. However the series inherits the divergent behavior for $\frac{a_1 k_0^2}{\nu_0^2} < -1$ due to the singularity at $\frac{a_1 k_0^2}{\nu_0^2} = 1$. For any value of $\frac{a_1 k_0^2}{\nu_0^2}$ outside the unit sphere centered at the origin the series will diverge due to that same singularity.

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