Nonlinear inversion of absorptive/dispersive wave field measurements: preliminary development and tests

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Abstract

We consider elements of the nonlinear inversion of primaries with a particular emphasis on the viscoacoustic, or absorptive/dispersive case. Since the main ingredient to higher-order terms in the inverse scattering series (as it is currently cast) is the linear, or Born, inverse, we begin by considering its "natural" form, i.e. without the corrections and/or assumptions discussed in Innanen and Weglein (2004). The absorptive/dispersive linear inverse for a single interface is found to be a filtered step-function, and for a single layer a set of increasingly smoothed step-functions. The "filters" characterizing the model shape are analyzed.

We next consider the nature of the inversion subseries, which would accept the linear inverse output and via a series of nonlinear operations transform it into a signal with the correct amplitudes (i.e. the true Q profile). The nature of the data event communication espoused by these nonlinear operations is discussed using a simple 1D normal incidence physical milieu, and then we focus on the viscous (attenuating) case. We show that the inversion subseries correctly produces the Q profile if the input signal has had the attenuation (or viscous propagation effects) compensated. This supports the existing ansatz (Innanen and Weglein, 2003) regarding the nature of a generalized *imaging* subseries (i.e. not the inversion subseries) for the viscous case, as carrying out the task of Q compensation.

1 Introduction

The linear viscoacoustic inversion procedure of Innanen and Weglein (2004), including the "pure" linear inversion and the subsequent "patches" used to correct for transmission error, may well be of practical use. The authors show that, numerically, the linear Q estimate is close to the true value in cases where strong nonlinear effects (such as propagation in the non-reference medium) are not present. However in any real situation the nonlinear nature of the relationship between the data and the parameters of interest will be present. Patching the procedure, through the use of intermediate medium parameter estimates, to correct for nonlinearity has been shown to be effective in numerical tests. However, this relies on *linear* parameter and operator estimates, and hence error will accumulate; furthermore, the application is ad hoc, and in any real medium the analogous procedure would require medium assumptions.

The linear inverse output is important on its own (especially when constructed using judiciously chosen – low – frequency inputs), but also as the potential input to higher order terms in the series. In either case, the quality and fidelity of the linear inversion results are of great importance; what defines those qualities, however, may differ. For instance, consider the imaging subseries of the inverse scattering series (Weglein et al., 2002; Shaw et al., 2003). Its task is to take the linearized inversion, which consists of contrasts that (*i*) are incorrectly located, and (*ii*) are of the incorrect amplitude (both in comparison to the true model), and correct the locations of the contrasts without altering the amplitudes. This task is accomplished, in 1D normal incidence examples, by weighting the derivatives of the linearized inversion result by factors which rely on the *incorrect* (Born) amplitudes. In other words, the imaging subseries relies on the correct provision of the incorrect model values. It doesn't matter that the linear inverse is not close to the true inverse – the inverse scattering series expects this. What matters is that the incorrect linear results are of high fidelity, so that a sophisticated nonlinear inversion procedure may function properly to correct them.

The full inverse scattering series provides a means to carry out viscoacoustic/viscoelastic inversion, in the absence of assumptions about the structure of the medium, and without ad hoc and theoretically inconsistent corrections and patches. This paper addresses some preliminary issues regarding the posing of the nonlinear inverse problem for absorptive/dispersive media, including a discussion and demonstration of the "natural" form for the linear inverse result – important because the linear inverse result is the input for all higher order terms in the various subseries as they are currently cast. We also consider some general aspects of the inversion subseries from the standpoint of the nonlinear, corrective communication between events that is implied by the computation of the terms in this subseries. We finally show results of the inversion subseries performing a nonlinear Q-estimation, *but using a Q-compensated data set*, and comment on the implications for nonlinear absorptive/dispersive inversion.

2 Input Issues: What Should a Viscous V_1 Look Like?

Second- and higher-order terms in the inverse scattering series will work to get around the problems of the linear inverse in a theoretically consistent way, one that does not require the linearly solved-for model to be close to the true model. We need to understand what these nonlinear methods expect as input. The subseries use *structure* and *amplitude* information in the Born approximation to construct the desired subsurface model; what then of the step-like structural assumptions we made previously? They are unlikely to be appropriate. What we need is to postulate the "natural" form of the linear viscoacoustic inverse, the form expected by the higher order terms in both subseries.

The "patch-up" structural form for the linear inversion of the 1D normal incidence problem was adopted to permit two parameters to be solved-for. We can avoid this assumption and investigate the natural linear inverse form by considering a two parameter-with-offset inversion, as discussed in Innanen and Weglein (2004), or a linear single parameter inversion from data associated with contrasts in Q only. For the sake of parsimony, we consider the latter here.

2.1 Viscous V_1 for a Single-Interface

Consider first the single interface configuration illustrated in Figure 1. The data reflected from such a scatterer is

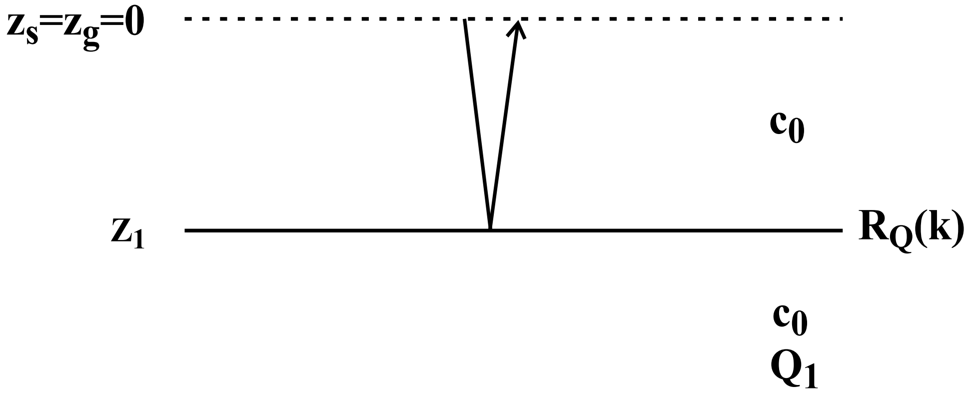

Figure 1: *Single interface experiment involving a contrast in Q only.*

$$D(k) = R_Q(k)e^{i2kz_1}, \tag{1}$$

where as ever, $k = \omega/c_0$, z_1 is the depth of the Q contrast, and the source and receiver are coincident at $z_s = z_g = 0$. Using terminology developed in Innanen and Weglein (2004), the reflection coefficient is

$$R_Q(k) = -\frac{F(k)}{Q_1\left[2 + \frac{F(k)}{Q_1}\right]}. \tag{2}$$

An example data set is illustrated in Figure 2. General 1D normal incidence data equations for wavespeed perturbations (α_1) and Q perturbations (β_1) were produced by Innanen and Weglein (2004):

$$\alpha_1(-2k) - 2\beta_1(-2k)F(k) = 4\frac{D(k)}{i2k}, \tag{3}$$

which can be altered to serve our current purposes by setting $\alpha_1(-2k) \equiv 0$. With this alteration we have a single equation and a single unknown for each wavenumber k, and thus a well-posed if not overdetermined problem. The revised equations are

$$\beta_1(-2k) = -2\frac{D(k)}{i2kF(k)}. \tag{4}$$

We can gain some insight by inserting the analytic form for data from a single interface case (equation 1):

$$\beta_1(-2k) = -2\frac{R_Q(k)e^{i2kz_1}}{i2kF(k)} = -2\frac{R_Q(k)}{F(k)}\frac{e^{i2kz_1}}{i2k}. \tag{5}$$

Recognizing the frequency-domain form for a Heaviside function, and making use of the convolution theorem, the linear form for $\beta_1(z)$ in the pseudo-depth z domain is

$$\beta_1(z) = L_1(z) * H(z - z_1), \tag{6}$$

where $*$ denotes convolution, and $L_1(z)$ is a spatial filter of the form

$$L_1(z) = \int_{-\infty}^{\infty} e^{-i2kz} L_1(-2k) d(-2k), \tag{7}$$

in which

$$L_1(-2k) = \frac{2R_Q(k)}{F(k)} = \frac{2}{2Q_1 + \left[\frac{i}{2} - \frac{1}{\pi}\ln\left(\frac{k}{k_r}\right)\right]}, \tag{8}$$

the latter arising from the substitution of the reflection coefficient. From the point of view of equation (6), the linear inverse output will be a Heaviside function altered (filtered) by a function that depends on the reflection coefficient and $F(k)$; obviously, the more this filter $L_1(z)$ differs from a delta function, the more the linear absorptive/dispersive reconstruction will be perturbed away from the (correct) step-like form. Figures 3 – 5 illustrate this combination and the resulting form for $\beta_1(z)$ for several values of Q_1.

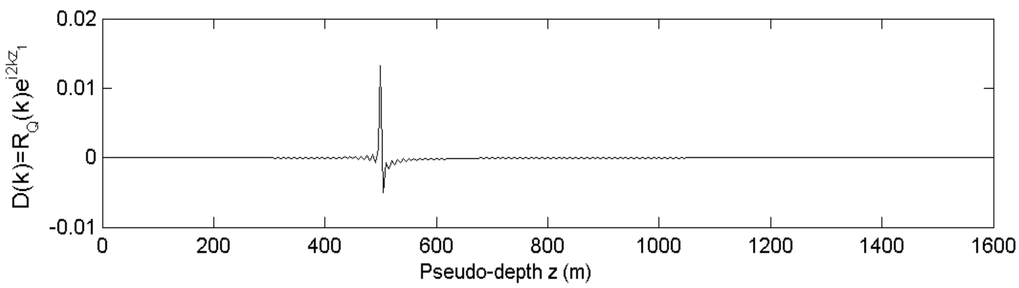

Figure 2: *Example numerical data for a single interface experiment with acoustic reference/viscoacoustic non-reference contrast.*

At large Q contrast (i.e. Figure 5) there is a distinct effect on the spatial distribution of the recovered linear perturbation. Nevertheless, compared to the result that would be produced by simply integrating the data $D(k)$ (in which there is a strong filtering $R_Q(k)$ upon what would usually be a delta-like event) the effect is small. In other words, there has been an attempt by the linear inverse formalism to correct for the filtering effects of $R_Q(k)$. By inspection of equation (4), the correction takes the form of spectral division of the data $D(k)$ by $F(k)$.

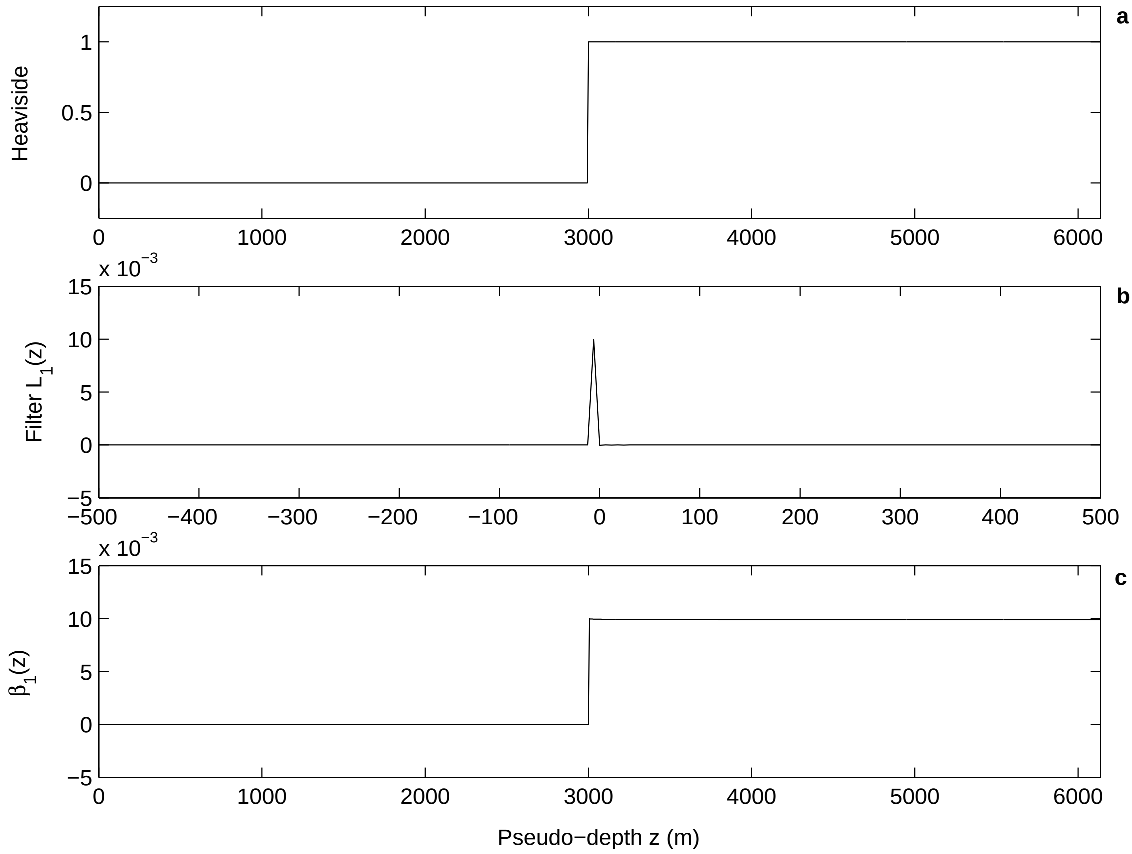

Figure 3*: Recovered single parameter viscoacoustic model,* $Q_1 = 100$*: (a) Heaviside component with step at* z_1*, (b) Filter* $L_1(z)$ *to be convolved with Heaviside component, (c)* $\beta_1(z) = L_1(z) * H(z - z_1)$*. At this contrast level,* $L_1(z)$ *is close to a delta-function and there is little effect other than a scaling of the Heaviside function.*

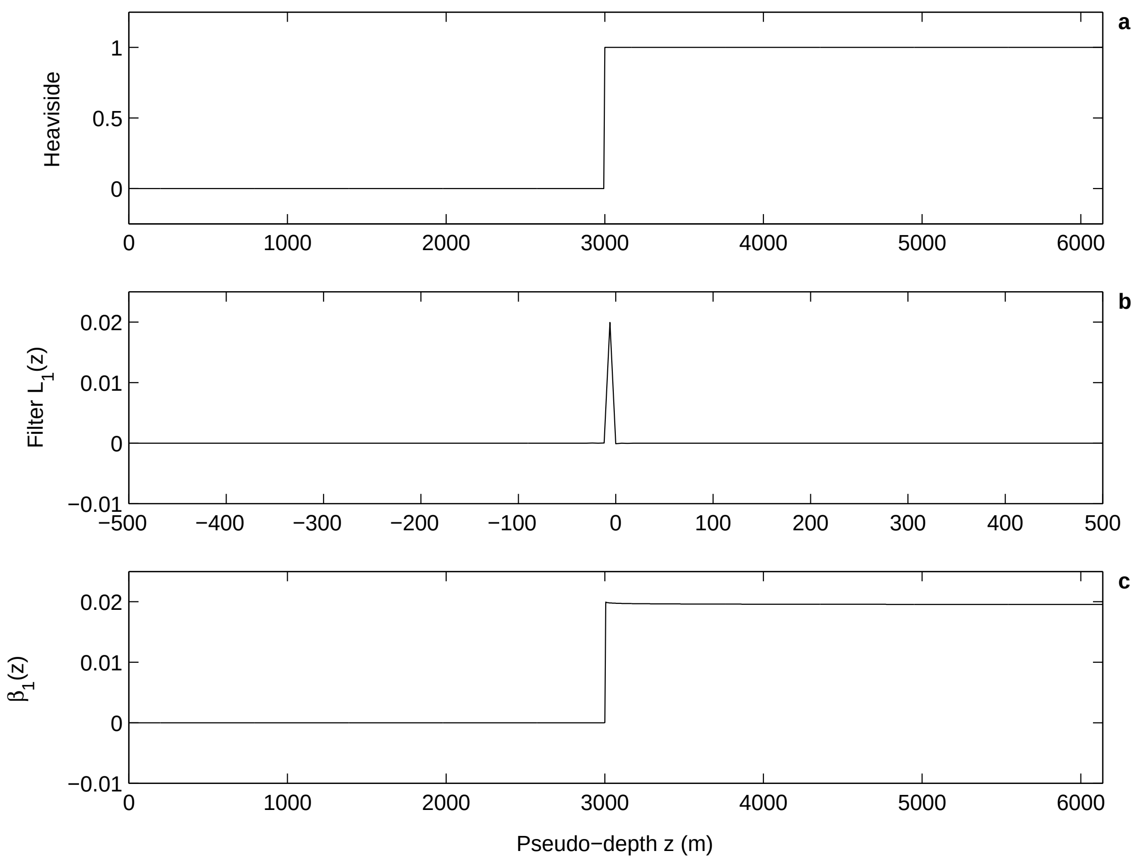

Figure 4*: Recovered single parameter viscoacoustic model, $Q_1 = 50$: (a) Heaviside component with step at z_1, (b) Filter $L_1(z)$ to be convolved with Heaviside component, (c) $\beta_1(z) = L_1(z) * H(z - z_1)$. At this contrast level, there is a slight visible effect on the Heaviside, including a slight "droop" with depth.*

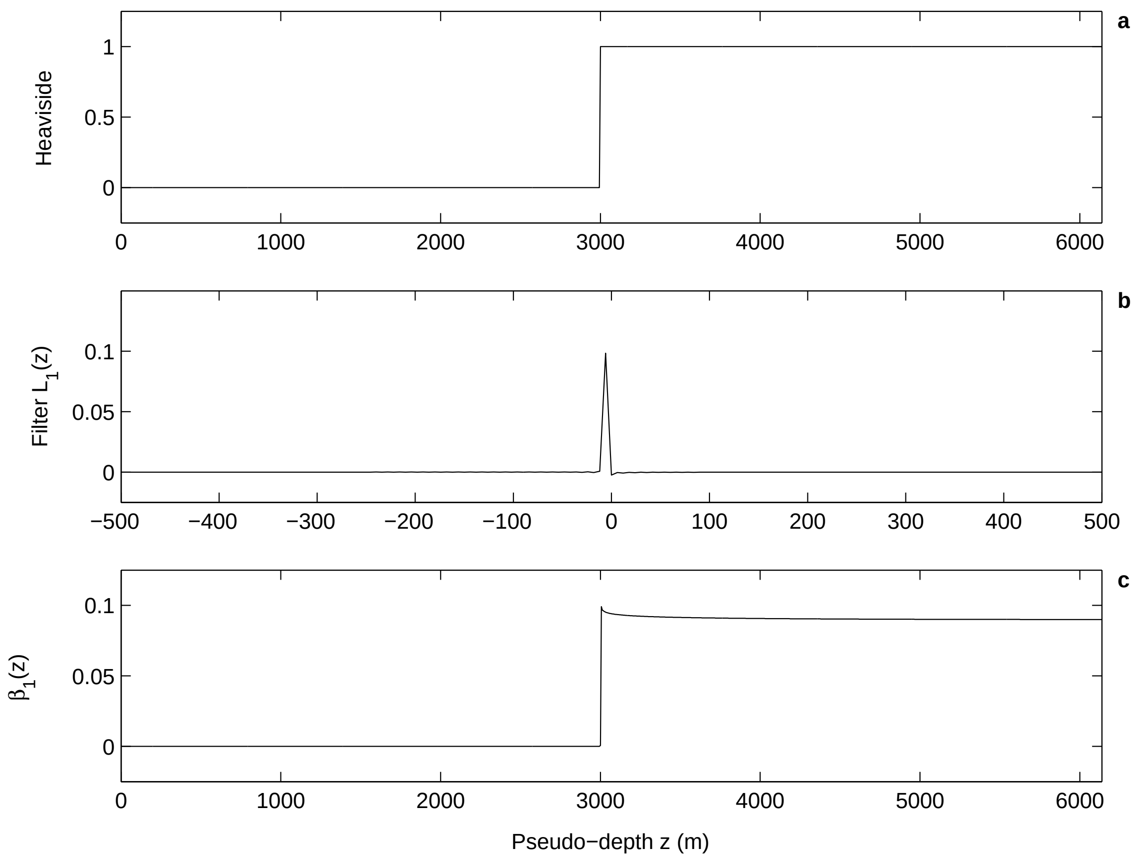

Figure 5*: Recovered single parameter viscoacoustic model,* $Q_1 = 10$*: (a) Heaviside component with step at* z_1*, (b) Filter* $L_1(z)$ *to be convolved with Heaviside component, (c)* $\beta_1(z) = L_1(z) * H(z - z_1)$*. At this contrast level,* $L_1(z)$ *produces a noticeable effect on the spatial structure of the recovered parameter distribution.*

2.2 Viscous V_1 for Multiple-Interfaces

The linear inversion for a single-interface $\beta_1(z)$ produces a filtered Heaviside function whose spectrum is an imperfect correction of the first integral of the viscoacoustic reflection coefficient. Numerical examples indicate that the correction is good up to high Q contrast, after which a distinct impact on the recovered spatial distribution of $\beta_1(z)$ is noticed. Any deeper reflectors will produce data events which are likewise determined by absorptive/dispersive reflectivity, but more importantly will have been affected by viscous propagation.

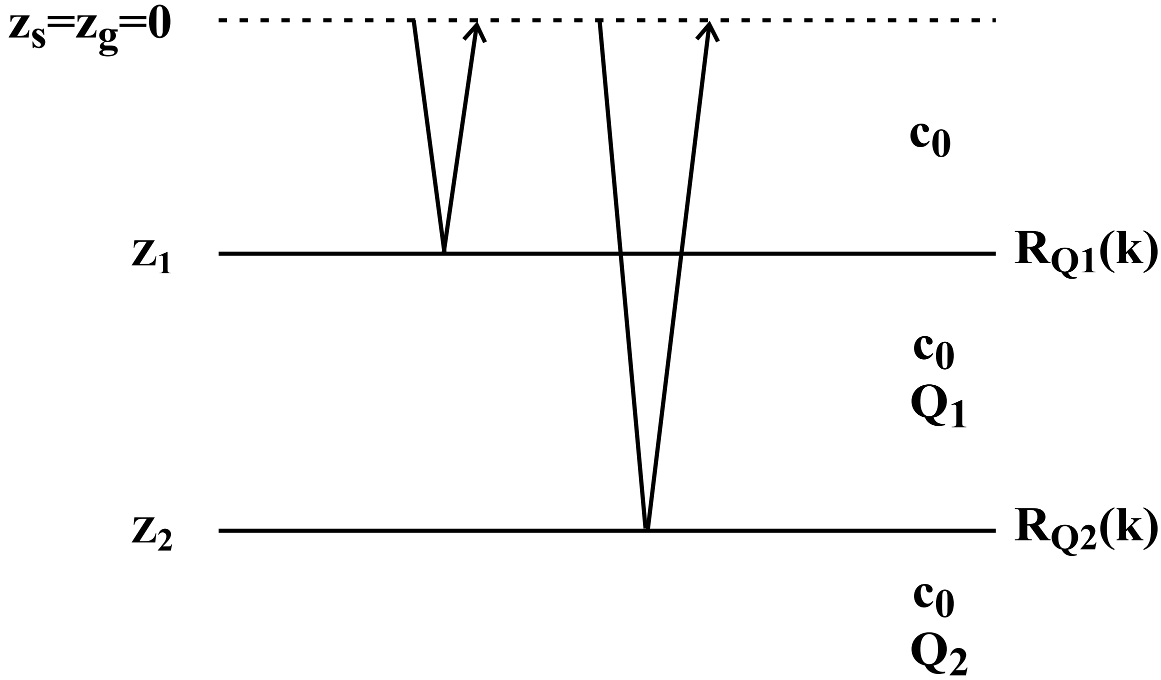

Figure 6*: Single layer experiment involving contrasts in Q only.*

Consider the single viscoacoustic layer example of Figure 6, which produces data qualitatively like the example in Figure 7. The data from such an experiment has the form

$$D(k) = R_{Q1}(k)e^{i2kz_1} + R'_{Q2}(k)e^{i2kz_1}e^{i2k_1(z_2-z_1)}, \tag{9}$$

where k_1 is the propagation wavenumber within the layer, generally for this example

$$k_j = \frac{\omega}{c_0}\left[1 + \frac{F(k)}{Q_j}\right], \tag{10}$$

(notice that the wavespeed remains c_0), and the reflection coefficient is

$$R'_{Q2}(k) = [1 - R_{Q1}(k)]^2\frac{k_1 - k_2}{k_1 + k_2} = [1 - R_{Q1}(k)]^2\,\frac{F(k)\left[\frac{1}{Q_1} - \frac{1}{Q_2}\right]}{2 + F(k)\left[\frac{1}{Q_1} + \frac{1}{Q_2}\right]}. \tag{11}$$

By separating the "ballistic" component of k_1 from the absorptive/dispersive component, the data may be re-written as

$$D(k) = [R_{Q1}(k)]\, e^{i2kz_1} + \left[R'_{Q2}(k) e^{i2k\frac{F(k)}{Q_1}(z_2-z_1)}\right] e^{i2kz_2}. \tag{12}$$

We can again gain some insight into the form of $\beta_1(z)$ by substituting in the analytic data in equation (12):

$$\begin{aligned}\beta_1(-2k) &= -2\frac{D(k)}{i2kF(k)}\\ &= \left[-\frac{2R_{1Q}(k)}{F(k)}\right]\frac{e^{i2kz_1}}{i2k} + \left[-\frac{2R'_{Q2}(k)e^{i2k\frac{F(k)}{Q_1}(z_2-z_1)}}{F(k)}\right]\frac{e^{i2kz_2}}{i2k}\\ &= L_1(-2k)\frac{e^{i2kz_1}}{i2k} + L_2(-2k)\frac{e^{i2kz_2}}{i2k}\end{aligned} \tag{13}$$

where

$$\begin{aligned}L_1(-2k) &= -\frac{2R_{1Q}(k)}{F(k)},\\ L_2(-2k) &= -\frac{2R'_{Q2}(k)e^{i2k\frac{F(k)}{Q_1}(z_2-z_1)}}{F(k)},\end{aligned} \tag{14}$$

are once again filters whose inverse Fourier transforms $L_1(z)$ and $L_2(z)$ act upon Heaviside functions:

$$\beta_1(z) = L_1(z) * H(z - z_1) + L_2(z) * H(z - z_2). \tag{15}$$

The second filter L_2 contains components that will impart effects of attenuative propagation onto the linear inverse result – see equation (14). Meanwhile, the only "processing" that occurs is the deconvolution of $F(k)$, which we have seen goes some way towards correcting for the phase/amplitude effects of the viscoacoustic reflectivity. The linear inversion will therefore include the unmitigated effects of propagation in its output.

Figures 8 – 9 illustrate the linear reconstruction of the two-event, single layer problem. Not surprisingly there is a characteristic smoothness – and error – in the linear inverse results below the first interface.

In summary, we may utilize a one-parameter, 1D normal incidence absorptive/dispersive linear inverse milieu to investigate the "natural" spatial form of the viscoacoustic Born approximation. This is useful in that (1) such a form is what the higher order terms of

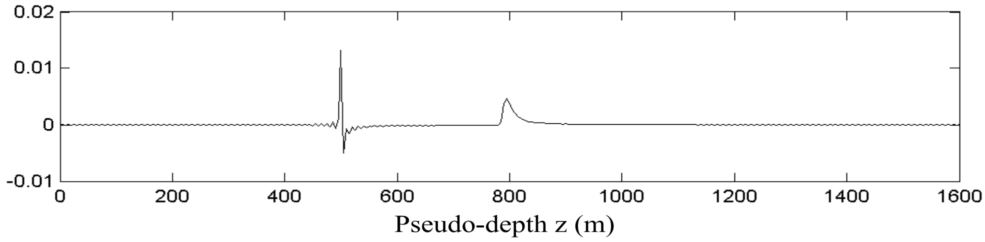

Figure 7: *Example numerical data for a single layer experiment with acoustic reference/viscoacoustic non-reference contrast.*

the series will be expecting – also, it is the normal-incidence version of the form we may expect from the multi-parameter, 1D-with offset inversion output developed theoretically by Innanen and Weglein (2004) – and (2) we see what remains to be done by the higher-order terms in the series.

The results are consistent with the statements of Innanen (2003) and Innanen and Weglein (2003); namely, that amplitude adjustment or [nonlinear] Q-estimation is required, and that depropagation, or nonlinear Q-compensation (in which the smooth edges of Figures 8 – 9 are sharpened) is also required. The balance of this paper is concerned with the former task; the latter is considered future work.

3 Nonlinear Event Communication in Inversion

At the same time that we consider specifically viscoacoustic inputs to the nonlinear inversion procedures afforded us by the inverse scattering series, we further consider some general aspects of the *inversion subseries* – i.e., those component terms of the series which devote themselves to the task of parameter identification. In particular, we comment in this section on the nature of the *communication between events* that occurs during the computation of the inversion subseries, and what that implies regarding the relative importance of nonlinear combinations of events.

Consider momentarily the relationship between a simple 1D normal incidence acoustic data set and its corresponding Born inverse. Imagine the data set to be made up of the reflected primaries associated with a single layer of wavespeed c_1, bounded from above by a half space of wavespeed c_0, and below by a half space of wavespeed c_2. If the interfaces are at $z_1 > 0$ and $z_2 > z_1$, and the source and receiver are at $z_s = z_g = 0$, the data is

$$D(k) = R_1 e^{i2\frac{\omega}{c_0}z_1} + R_2' e^{i2\frac{\omega}{c_0}z_1} e^{i2\frac{\omega}{c_1}(z_2-z_1)}, \tag{16}$$

where $R_2' = (1-R_1)^2 R_2$ contains the effects of shallower transmission effects, and R_1 and R_2 are the reflection coefficients of the upper and lower interfaces respectively. If c_0 is taken

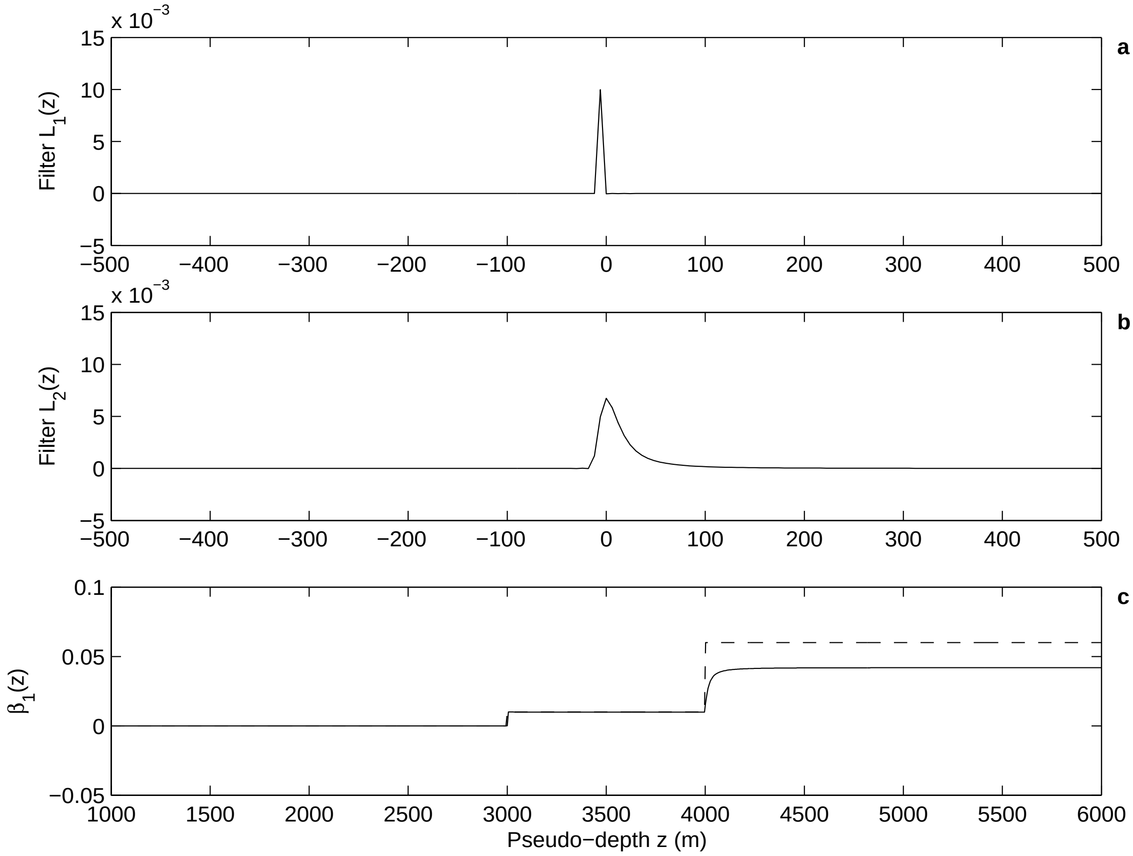

Figure 8: *Illustration of a recovered single parameter viscoacoustic model, $Q_1 = 100$, $Q_2 = 20$: (a) Filter $L_1(z)$ to be convolved with Heaviside component $H(z - z_1)$, (b) Filter $L_2(z)$ to be convolved with Heaviside component $H(z - z_2)$, (c) $\beta_1(z) = L_1(z) * H(z - z_1) + L_2(z) * H(z - z_2)$. The propagation effects on the estimate of the lower interface are apparent.*

to be the wavespeed everywhere, and pseudo-depths z'_1 and z'_2 are assigned to events of half the measured traveltime and this reference wavespeed assumption, equation (16) may be written

$$D(k) = R_1 e^{ikz'_1} + R'_2 e^{ikz'_2}, \tag{17}$$

where $k = \omega/c_0$. We now drop the primes in z for convenience, but remain in the pseudo-depth domain for the duration of this section. In the space domain this reads

$$D(z) = R_1\delta(z - z_1) + R'_2\delta(z - z_2). \tag{18}$$

Using standard terminology, the Born inverse associated with the above data set, $\alpha_1(z) = V_1(k, z)/k^2$, is given by

$$\alpha_1(z) = 4\int_0^z D(z'')dz'' = 4R_1 H(z - z_1) + 4R'_2 H(z - z_2). \tag{19}$$

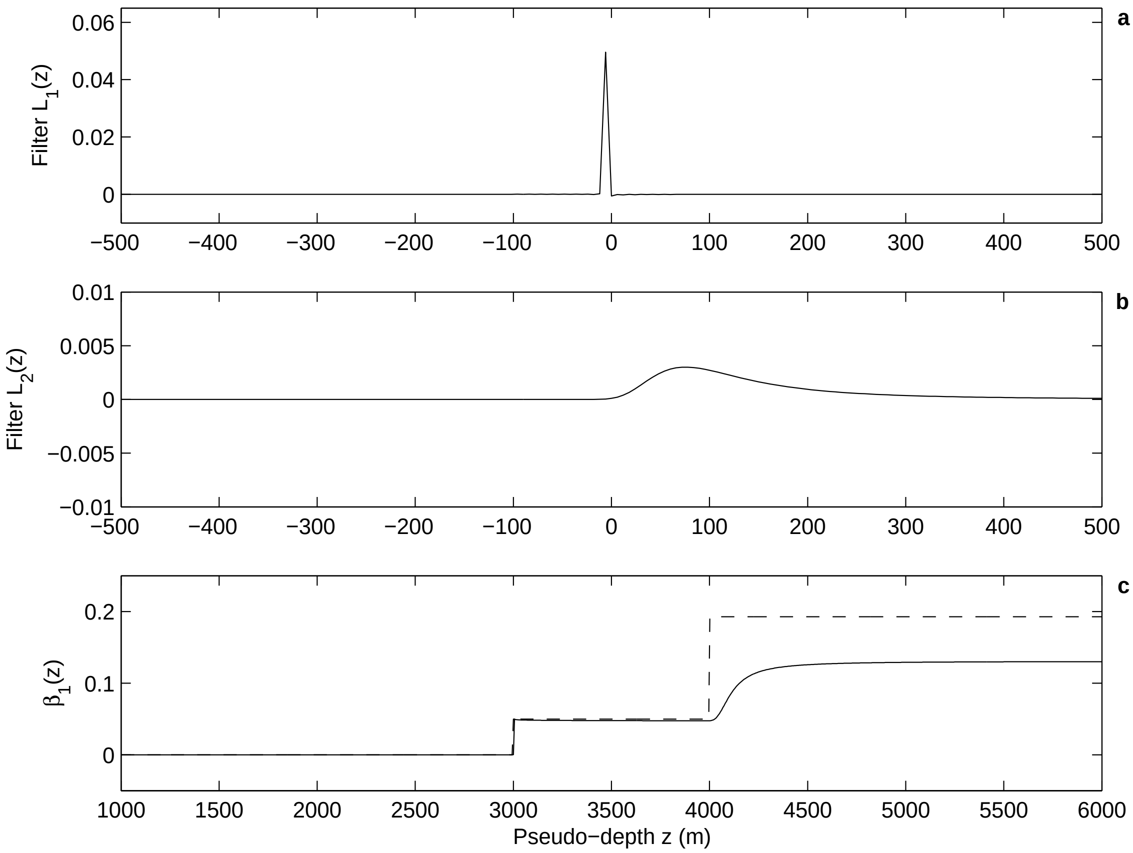

Figure 9*: Illustration of a recovered single parameter viscoacoustic model, $Q_1 = 20$, $Q_2 = 10$: (a) Filter $L_1(z)$ to be convolved with Heaviside component $H(z - z_1)$, (b) Filter $L_2(z)$ to be convolved with Heaviside component $H(z - z_2)$, (c) $\beta_1(z) = L_1(z) * H(z - z_1) + L_2(z) * H(z - z_2)$. The propagation effects on the estimate of the lower interface are increasingly destructive, and produce a characteristic smoothing.*

The inversion subseries then acts upon this quantity nonlinearly:

$$\alpha_{INV}(z) = \sum_{j=1}^{\infty} A_j \alpha_1^j(z), \tag{20}$$

(where $A_j = [j(-1)^{j-1}]/[4^{j-1}]$) i.e. by summing weighted powers of $\alpha_1(z)$.

Let us begin by considering the form of the Born inverse result of equation (19). The Heaviside sum is in some sense not a good way of expressing the *local* amplitude of $\alpha_1(z)$. Another way of writing it is

$$\alpha_1(z) = 4R_1 \left[H(z - z_1) - H(z - z_2)\right] + (4R_1 + 4R_2')H(z - z_2), \tag{21}$$

which now expressly gives the local amplitudes. This is somewhat more edifying in the sense that it explicitly makes reference to the linear (additive) communication between events implied by the linear inversion for the lower layer (i.e. R_1 and R_2'). Let us make this even more explicit with a definition. Let:

$$\alpha_1(z) = D_1\left[H(z-z_1) - H(z-z_2)\right] + (D_1+D_2)H(z-z_2'), \quad (22)$$

where $D_1 = 4R_1$, $D_2 = 4R_2'$, or more generally $D_n = 4R_n'$, the subscript referring explicitly to the event in the data that contributed to that portion of the inverse. Equation (22) straightforwardly generalizes to N layers, if we define $H(z-z_{N+1}) \equiv 0$:

$$\alpha_1(z) = \sum_{n=1}^{N}\left(\sum_{i=1}^{n} D_i\right)[H(z-z_n) - H(z-z_{n+1}]. \quad (23)$$

Figure 10 illustrates these definitions for the two-layer case. The Heavisides disallow interaction between the terms in brackets $(\cdot)$ (for different n values) under exponentiation, so

$$\alpha_1^j(z) = \sum_{n=1}^{N}\left(\sum_{i=1}^{n} D_i\right)^j [H(z-z_n) - H(z-z_{n+1})]. \quad (24)$$

This allows us the flexibility to consider the local behaviour of the inversion result $\alpha_{INV}(z)$ directly in terms of the data events D_k that contribute to it. The amplitude of the j'th term in the inversion subseries for the correction of the m'th layer is

$$A_j\left(\sum_{i=1}^{m} D_i\right)^j. \quad (25)$$

For the second layer, this amounts to

$$A_j\left(D_1+D_2\right)^j, \quad (26)$$

and for the third layer,

$$A_j\left(D_1+D_2+D_3\right)^j. \quad (27)$$

Notice that in terms of D_n, the computation of the inversion subseries will produce terms similar to the exponentiation of a multinomial. Therefore each term of the inversion subseries (in this simple 1D normal incidence case) will involve a set of subterms whose coefficients are given by the multinomial formula. For instance, the first three terms for the second layer are

$$\begin{aligned} &A_1(D_1+D_2),\\ &A_2(D_1^2+2D_1D_2+D_2^2),\\ &A_3(D_1^3+3D_1^2D_2+3D_1D_2^2+D_2^3), \end{aligned} \quad (28)$$

etc., i.e. they depend strongly on the familiar coefficients of the binomial formula:

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ 1\ \ 4\ \ 6\ \ 4\ \ 1 \\ \vdots \end{array} \tag{29}$$

Meanwhile the third layer follows trinomial-like terms, and so forth. This has an interesting consequence. Consider the third-order term of the third layer:

$$A_3(D_1^3+D_2^3+D_3^3+3D_1^2D_2+3D_1^2D_3+3D_2^2D_1+3D_2^2D_3+3D_3^2D_1+3D_3^2D_2+6D_1D_2D_3). \tag{30}$$

In equation (30): all else being equal (i.e. if we assume that events D_1, D_2, and D_3 are of approximately equal size), the term which combines all events evenly, $D_1D_2D_3$, can be seen to be six times as important as a term which involves a single event only, e.g. D_1^3. The multinomial formula therefore not only informs us about the distribution of communicating data events produced by the inversion subseries, but it also predicts the relative importance of one kind of communication over another. If we believe what we see in equation (30), nonlinear interaction terms *which use information as evenly-distributed as possible* from all data events above the layer of interest will dominate.

Since the binomial/multinomial formulas are used to model the relative probabilities of the outcomes of random experiments, the inter-event communication distribution problem is analogous to any number of well-known betting games, such as throwing dice or tossing coins. For instance, the relative probabilities of observing heads (h) vs. tails (t) in n coin tosses is given by the binomial formula:

$$(t+h)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} t^{n-j} h^j, \tag{31}$$

and the rolling of dice is similarly given by the multinomial formula for a 6-term generating function.

The inversion subseries is of course anything but random or stochastic. The point of the analogy is to show that, especially at high order, the contribution of evenly distributed nonlinear data communication is greater than that of any other type. For the two event case, the relative importance of even-distribution over biased distribution is precisely the same as the relative probability of tossing equal numbers of heads and tails over large numbers of either. In other words, the nonlinear inversion terms which simultaneously make use of as many data events as are available dwarfs any other contribution. This behaviour may be speaking of some intuitively reasonable feature of the inversion subseries, that in inverting for the amplitude of a deep layer, the global effect of what overlies it is far more important than any one (or few) local effect(s).

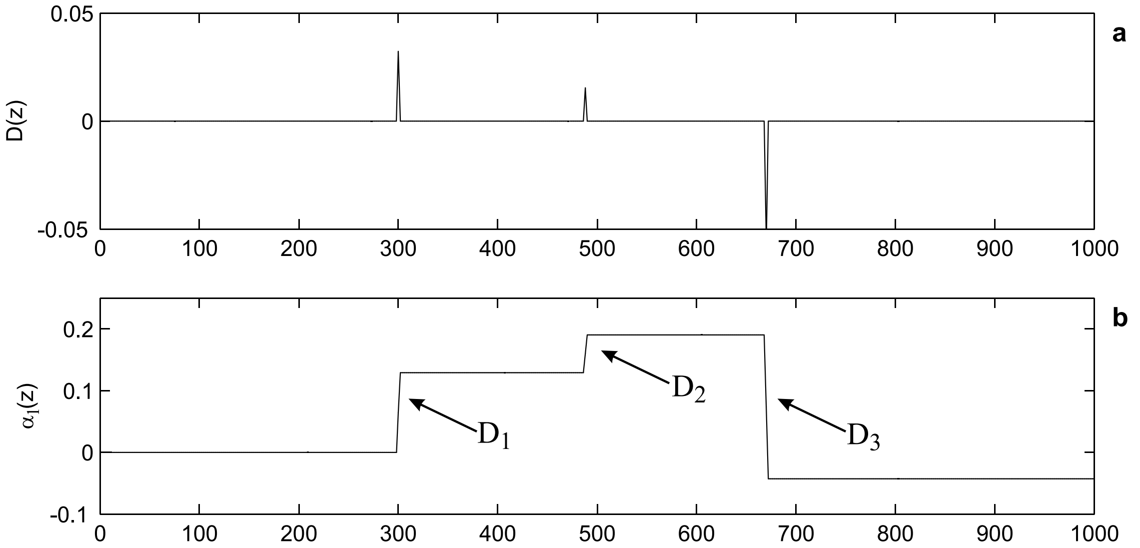

Figure 10: *Synthetic data (a) corresponding to a two-layer model; (b) the Born approximation $\alpha_1(z)$. Local increases or decreases in the linear inversion due to specific data events are labelled D_n.*

4 Nonlinear Q-estimation

In this section some properties of the inversion subseries as it addresses the reconstruction of attenuation contrasts from 1D seismic data are considered. A peculiar data set is created for this purpose, one that is designed not to reflect reality, but rather to examine certain aspects of the inverse scattering series and only those aspects. Also, it involves the choice of a simplified model of attenuation. We begin with a description of the physical framework of this analysis.

We use the following two dispersion relations:

$$\begin{aligned} k &= \frac{\omega}{c_0}, \\ k(z) &= \frac{\omega}{c_0}[1 + i\beta(z)], \end{aligned} \tag{32}$$

the first being that of the reference medium, and the second that of a non reference medium in which the wavespeed may vary, as may the parameter β, which, like the wavespeed is considered an inherent property of the medium in which the wave propagates. It is responsible for not only altered reflection and transmission properties of a wave, but also for the amplitude decay associated with intrinsic friction. One could argue pretty reasonably that β is related to the better known parameter Q by $\beta = 1/2Q$ because of their similar roles in changing the amplitude spectrum of an impulse. However, most Q models are more sophisticated in that they utilize dispersion to ensure the causality of the medium response. We make use of this “friction” model, which captures much of the key behaviour of attenuating media, to take advantage of its simplicity.

For this investigation, we restrict attention to models which contain variations in β only. Waves propagate with wavespeed c_0 everywhere.

4.1 A Further Complex Reflection Coefficient

Contrasts in β only will produce reflections with strength

$$R_n = \frac{k_{n-1} - k_n}{k_{n-1} + k_n} = \frac{i(\beta_{n-1} - \beta_n)}{2 + i(\beta_{n-1} + \beta_n)}, \tag{33}$$

and, since we assume an acoustic reference medium ($\beta_0 = 0$),

$$R_1 = \frac{-i\beta_1}{2 + i\beta_1}. \tag{34}$$

As in the dispersive models used elsewhere in this paper, the reflection coefficient is complex. Without dispersion, the complex reflection coefficient implies the combination of a frequency independent phase rotation and a real reflection coefficient: $R_1 = |R_1|e^{i\theta}$. This is akin to the reflection coefficients discussed in Born and Wolf (1999).

4.2 A Non-attenuating Viscoacoustic Data Set

A further simplification has to do with the measured data, rather than the medium itself. As in the acoustic case, true viscoacoustic data will be a series of transient pulses that return, delayed by an appropriate traveltime and weighted by a combination of reflection and (shallower) transmission coefficients. But unlike an acoustic medium, the propagation *between contrasts* also determines the character of the events. Each pulse that has travelled through a medium with $\beta \neq 0$ will have had its amplitude suppressed at a cycle-independent rate, and therefore be broadened. Although this is a fundamental aspect of viscoacoustic wave propagation (and indeed one of the more interesting aspects for data processing), for the purposes of this investigation we suppress it in creating the data. The results to follow will help explain why.

The perturbation γ is based on an attenuative propagation law, and is itself complex:

$$\gamma(z) = 1 - \frac{k^2(z)}{k_0^2} = -2i\beta(z) + \beta^2(z), \tag{35}$$

If the propagation effects are suppressed (or have been corrected for already), then the Born approximation of $\gamma(z)$, namely $\gamma_1(z)$ is

$$\gamma_1(z) = A_1 H(z - z_1) + A_2 H(z - z_2) + ..., \tag{36}$$

where the A_n are given by

$$A_n = 4R_n \prod_{j=1}^{n-1} (1 - R_j^2), \tag{37}$$

and the reflection coefficients $R_j(k)$ are given by equations (33) and (34).

In the previous section it was noted that the contributing terms to any layer in the inversion subseries can be found by invoking the multinomial formula; and the contributing terms to a two-layer model are produced by the binomial formula (which has a much more simply computable form). We assume a two-layer model, in which the reference medium ($c = c_0$, $\beta = 0$) is perturbed below z_1 such that $c = c_0$ and $\beta = \beta_1$, and again below z_2 such that $c = c_0$ and $\beta = \beta_2$. The depths z_1 and z_2 are chosen to be 300 m and 600 m respectively. This results in

$$\gamma_1(z) = \left[\frac{-4i\beta_1}{2 + i\beta_1}\right] H(z - z_1) + \left[\frac{-4i(\beta_1 - \beta_2)}{2 + i(\beta_1 + \beta_2)} \left(1 - \frac{\beta_1^2}{(2 + i\beta_1)^2}\right)\right] H(z - z_2). \tag{38}$$

The n'th power of the Born approximation for either layer of interest is computable by appealing to the binomial formula. The result of the inversion subseries on this two interface model is therefore

$$\gamma_{INV}(z) = A_1^{INV} H(z - z_1) + A_2^{INV} H(z - z_2), \tag{39}$$

where

$$\begin{aligned} A_1^{INV} &= \sum_{j=1}^{\infty} \frac{j(-1)^{j-1}}{4^{j-1}} A_1^j, \\ A_2^{INV} &= \sum_{j=1}^{\infty} \frac{j(-1)^{j-1}}{4^{j-1}} \left(\sum_{k=1}^{j} \frac{j!}{k!(j-k)!} A_1^{j-k} A_2^k \right), \end{aligned} \tag{40}$$

and where

$$A_1 = \frac{-4i\beta_1}{2+i\beta_1},$$
$$A_2 = \frac{-4i(\beta_1-\beta_2)}{2+i(\beta_1+\beta_2)}\left(1-\frac{\beta_1^2}{(2+i\beta_1)^2}\right) \tag{41}$$

are the amplitudes of the Born approximation in equation (36). Figures 11 – 13 demonstrate numerically the convergence of these expressions (the real part, for the sake of illustration) towards the true value. There is clear oscillation over the first three or so terms for the second layer, but the oscillations settle down to close to the correct value with reasonable speed. This suggests an important aspect of going beyond the Born approximation in Q estimation – the deeper layers, whose amplitudes are related not only to their own Q value but to those above them, require the invocation of several nonlinear orders to achieve their correct value.

Finally, Figures 14 – 16 illustrate the same inversion results as those seen in Figures 11 – 13, but in the context of the spatial distribution of the reconstructed perturbation.

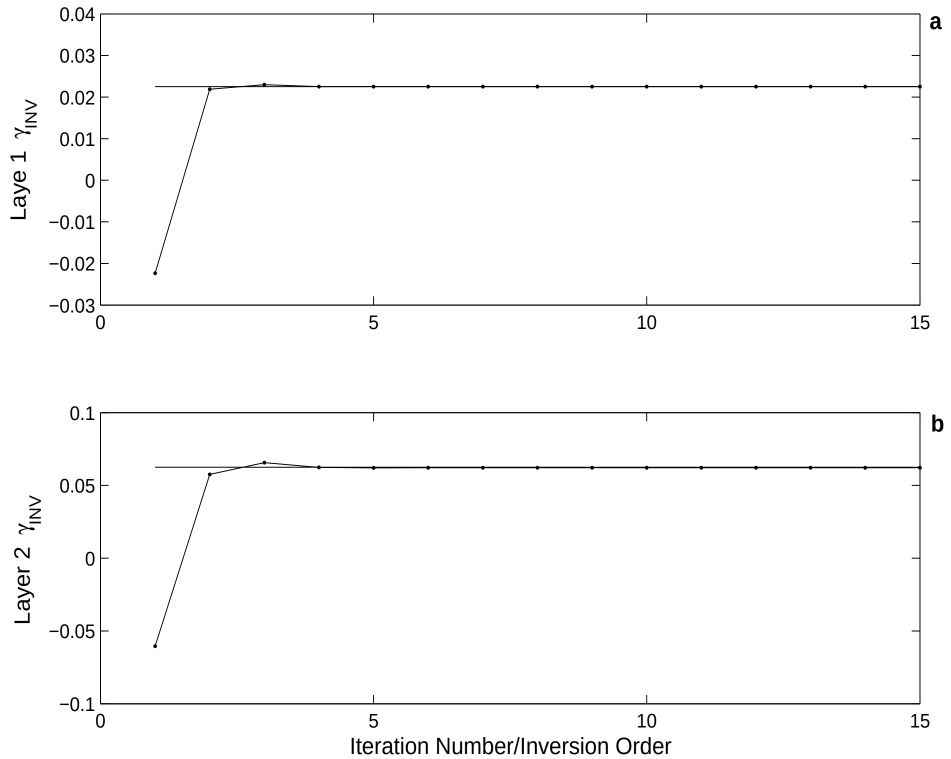

Figure 11: *Convergence of the viscoacoustic inversion subseries for a two-interface model, consisting of contrasts in (friction-model) attenuation parameter. The real component of the perturbation is plotted against inversion order or iteration number (connected dots) and with respect to the true perturbation value (solid). (a) layer* 1 *inversion. (b) layer* 2 *inversion. For model contrasts* $\beta_1 = 0.15$, $\beta_2 = 0.25$, $\gamma_{INV}(z)$ *is close to* $\gamma(z)$ *for both layers.*

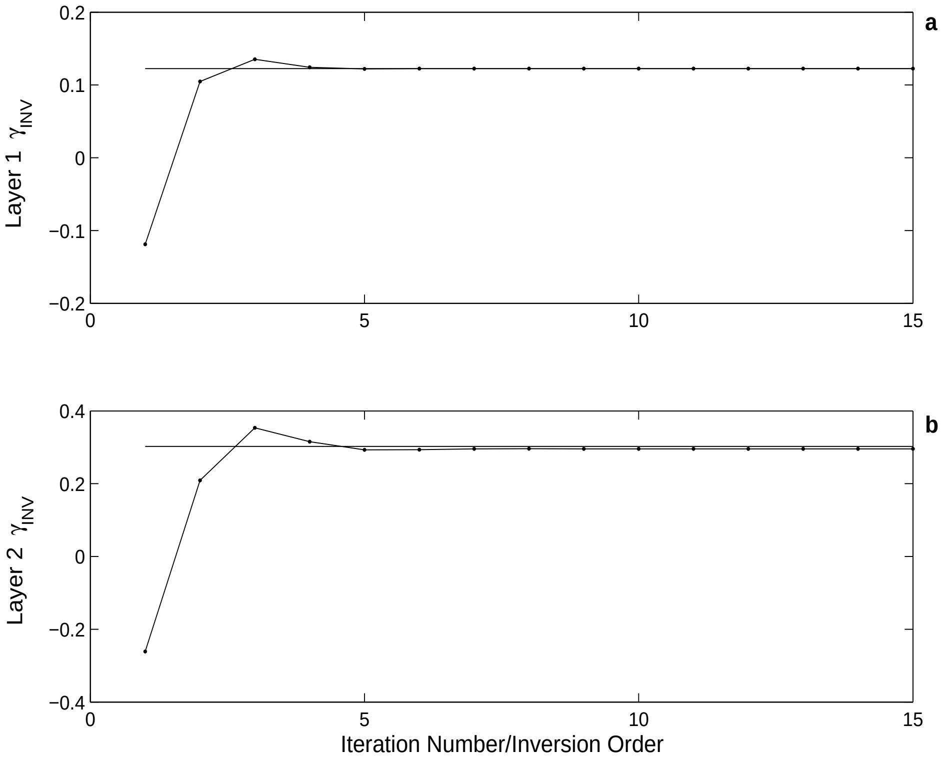

Figure 12*: Convergence of the viscoacoustic inversion subseries for a two-interface model, consisting of contrasts in (friction-model) attenuation parameter. The real component of the perturbation is plotted against inversion order or iteration number (connected dots) and with respect to the true perturbation value (solid). (a) layer* 1 *inversion. (b) layer* 2 *inversion. For model contrasts* $\beta_1 = 0.35$, $\beta_2 = 0.55$, $\gamma_{INV}(z)$ *differs slightly from* $\gamma(z)$ *in the lower layer.*

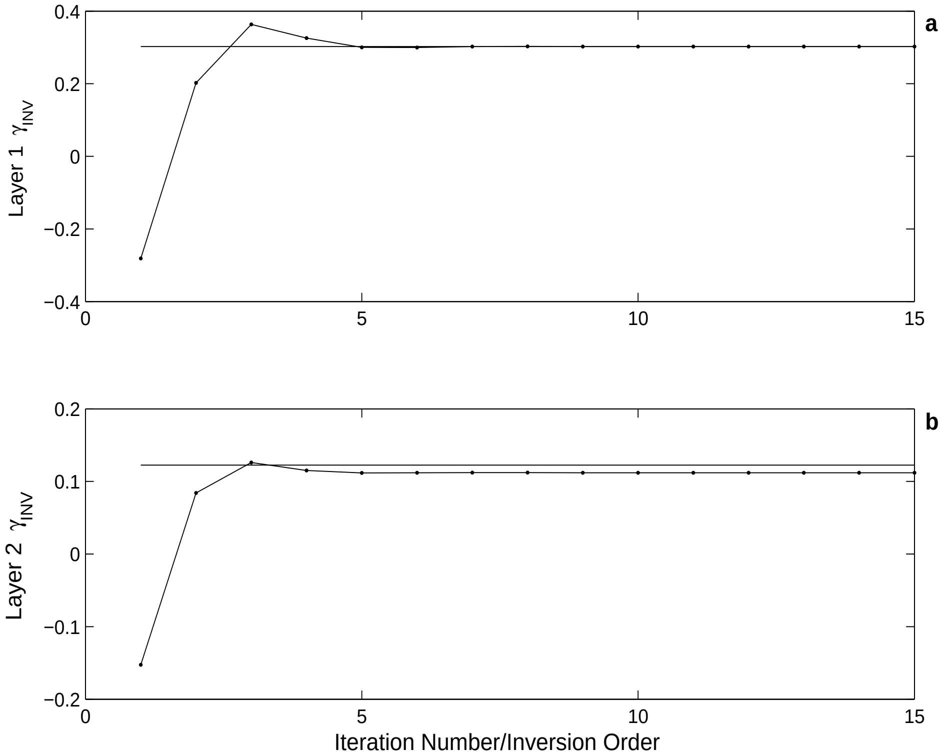

Figure 13*: Convergence of the viscoacoustic inversion subseries for a two-interface model, consisting of contrasts in (friction-model) attenuation parameter. The real component of the perturbation is plotted against inversion order or iteration number (connected dots) and with respect to the true perturbation value (solid). (a) layer* 1 *inversion. (b) layer* 2 *inversion. For model contrasts* $\beta_1 = 0.55$, $\beta_2 = 0.35$, $\gamma_{INV}(z)$ *again differs slightly from* $\gamma(z)$ *in the lower layer.*

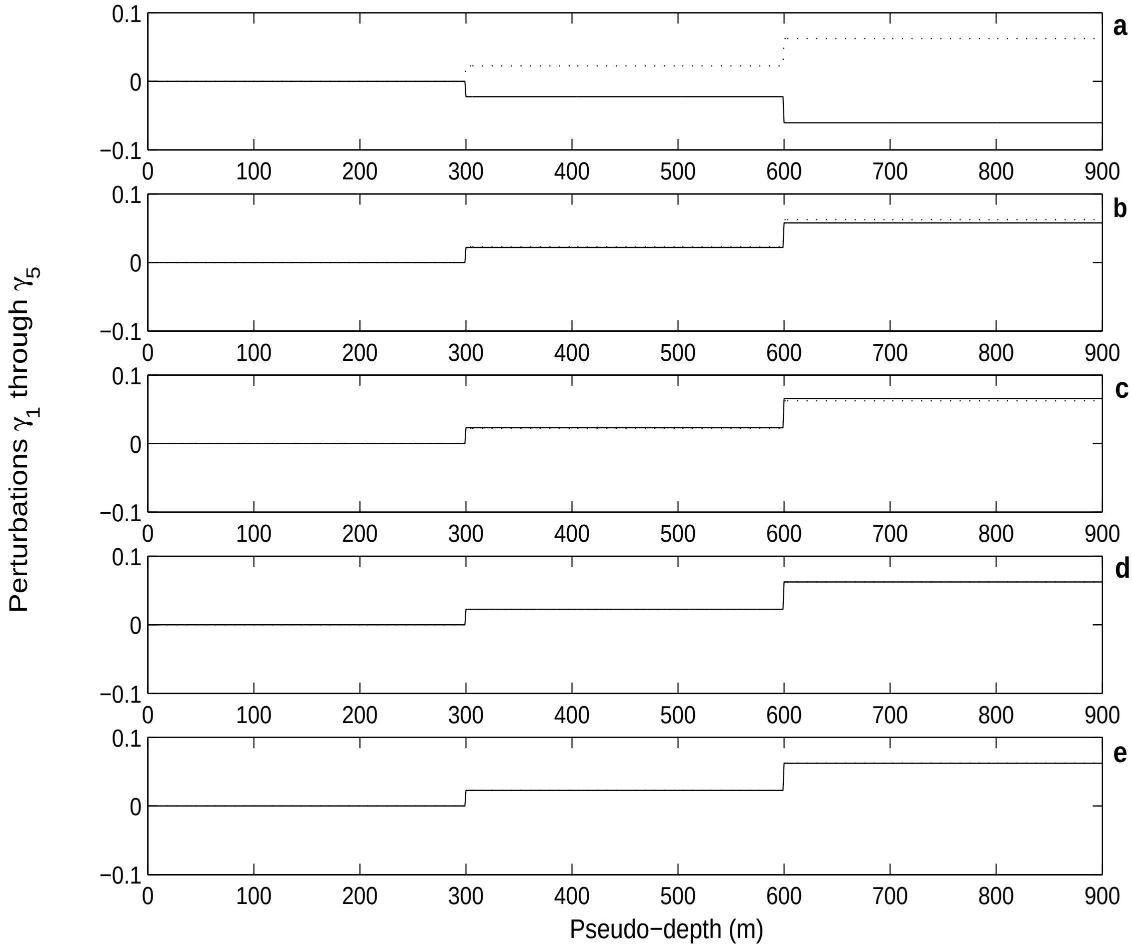

Figure 14: *Convergence of the viscoacoustic inversion subseries for a two- interface model, consisting of contrasts in (friction-model) attenuation parameter. Depth profile of the real component of the constructed perturbation (solid) is plotted against the true perturbation (dotted) for* 5 *iterations. (a) iteration* 1 *(Born approximation) – (e) iteration* 5. *Model inputs:* $\beta_1 = 0.15$, $\beta_2 = 0.25$.

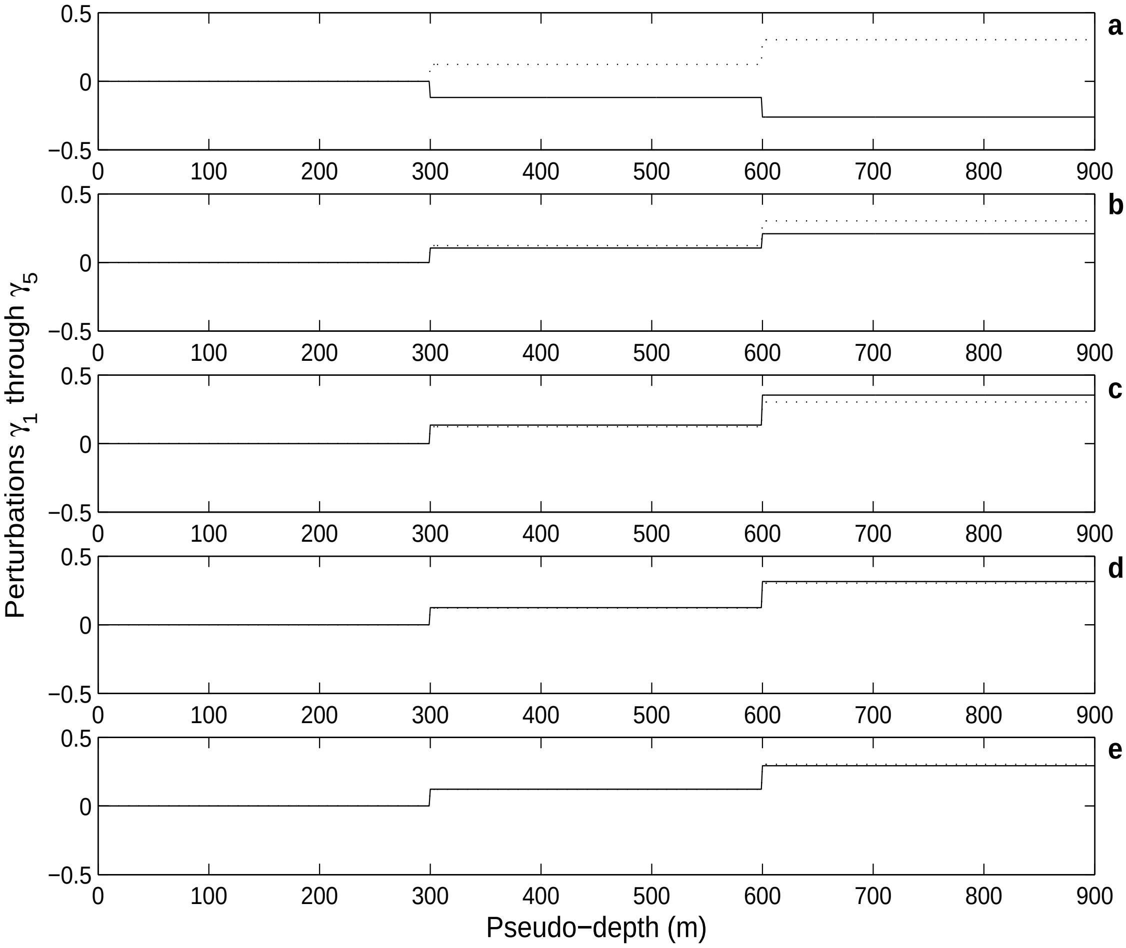

Figure 15: *Convergence of the viscoacoustic inversion subseries for a two- interface model, consisting of contrasts in (friction-model) attenuation parameter. Depth profile of the real component of the constructed perturbation (solid) is plotted against the true perturbation (dotted) for* 5 *iterations. (a) iteration* 1 *(Born approximation) – (e) iteration* 5*. Model inputs:* $\beta_1 = 0.35$, $\beta_2 = 0.55$.

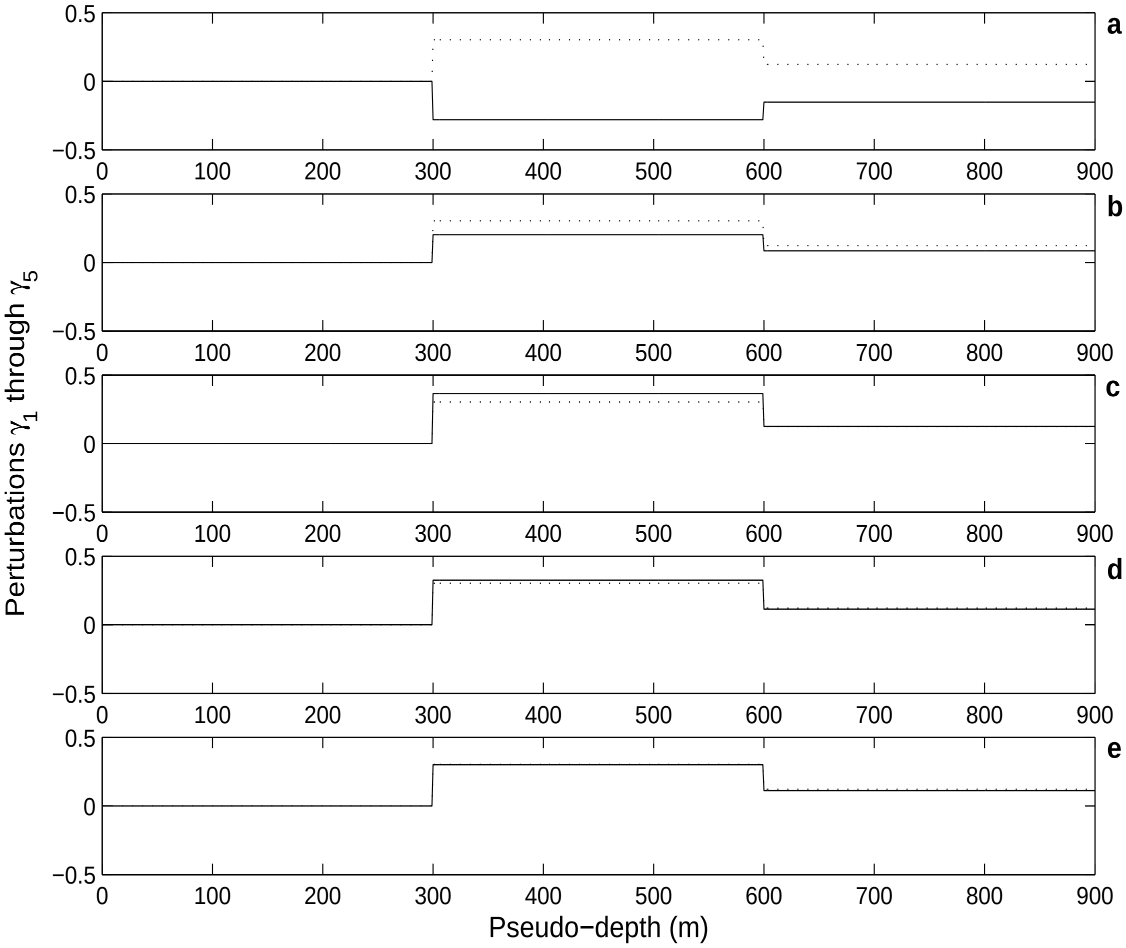

Figure 16*: Convergence of the viscoacoustic inversion subseries for a two- interface model, consisting of contrasts in (friction-model) attenuation parameter. Depth profile of the real component of the constructed perturbation (solid) is plotted against the true perturbation (dotted) for* 5 *iterations. (a) iteration* 1 *(Born approximation) – (e) iteration* 5*. Model inputs:* $\beta_1 = 0.55$*,* $\beta_2 = 0.35$*.*

The success of the inversion subseries in reconstructing the amplitudes of the attenuation contrasts in the examples of the previous section is worth pursuing, since the success was achieved by using a strange input data set; one in which the propagation effects of the medium had been stripped off *a priori*. How should this success be interpreted?

It follows that if the inversion subseries correctly uses the above data set to estimate Q – as opposed to one in which the propagation effects were present – then the subseries must have been expecting to encounter something of that nature.

The apparent conclusion is that the other non-inversion subseries terms should be looked-to to provide just such an attenuation-compensated Born approximation. This is in agreement with the conclusions reached by Innanen and Weglein (2003), in which it appeared that the terms representing the viscous analogy to the imaging subseries would be involved in the more general task of removing the effects of propagation (reflector location and attenuation/dispersion) on the wave field.

More generally, these results suggest that for these simple cases at least (1D normal incidence acoustic/viscoacoustic) we see distinct evidence that the inversion subseries produces corrected parameter amplitudes *when the output of the imaging/depropagation subseries is used as input.* This is a strong departure from the natural behavior of the inverse scattering series, which, of course, does not use output of one set of terms as input for another.

5 Conclusions

The aims of this paper have been to explore (1) the nature of the input to the nonlinear absorptive/dispersive methods implied by the mathematics of the inverse scattering series, and (2) the nature of the inversion subseries, or target identification subseries itself, both generally and with respect to the viscoacoustic problem.

Regarding the input, the viscoacoustic Born approximation is shown to have two characteristic elements – an amplitude/phase effect due to the reflectivity, and an absorptive/dispersive effect due to the propagation effects; the former is ameliorated somewhat in the linear inverse procedure, and the former is not. Hence the linear Q-profile is reconstructed with some account taken of the phase and amplitude spectra of the local events, but without any attempt to correct for the effective transmission associated with viscous propagation.

The nonlinear processing and inversion ideas fleshed out in this paper rely, in their detail, on specific models of absorption and dispersion. For instance, much of the formalism in the Born inversion for c and Q in the single interface case arises because the frequency dependence of the dispersion law ($F(k)$) is separable, multiplying a frequency-independent Q. Since the detail of the inversion relies on the good fortune of having a viscoacoustic model that behaves so accommodatingly, it is natural to ask: if we have tied ourselves to one particular model of dispersion, do we not then rise and fall with the (case by case) utility and accuracy of this one model? The practical answer is yes, of course; the specific formulae and methods in this paper require that this constant Q model explain the propagation and reflection/transmission in the medium. But one of the core ideas of this paper has been to root out the basic reasons why such an approach might work. The dispersion-based frequency dependence of the reflection coefficient as a means to separate Q from c is not a model dependent idea, regardless of model-to-model differences in detail and difficulty.

The more sophisticated Q model employed in this paper provided a complex, frequency-dependent scattering potential, which has since been used to approximate the Q profile in linear inversion, and to comment on the expected spatial form of the input to higher order terms. However, at an early stage in the derivation the form was simplified with a rejection of terms quadratic in Q. One might notice that in the latter portion of this paper the same step was *not* taken in the friction model perturbation that was used as input in the higher order terms. It is anticipated that the more sophisticated quadratic potential will be needed in the dispersive Q model when it is used as series input.

The results of this paper appear to confirm the predictions of the forward series analysis in Innanen and Weglein (2003); i.e. that prior to the inversion (target identification) step, the subseries terms which are nominally concerned with imaging, must also remove the effects of attenuative propagation. The confirmation in this paper is circumstantial but compelling: the ubiquitous increase in quality of the inverse results (both low order constant Q and high order friction-based attenuation) in the absence of propagation effects strongly suggest that depropagated (Q compensated) inputs are expected in these inversion procedures. A clear future direction for research is in generalizing the imaging subseries to perform Q compensation simultaneously with reflector location.

Acknowledgments

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