Initial analysis of the inverse scattering series for variable background

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Abstract

We present a derivation of the first two terms of the inverse scattering series that accommodate a smoothly varying background velocity. A WKBJ Green's function is assumed for reference propagation. Shaw et al. (2003) separate the second term in the series, using a constant reference medium, into contributions appropriate for the tasks of imaging and parameter estimation. We provide the generalization of that task separation for a variable background, $c_0(z)$.

1 Introduction

The current imaging and inversion sub-series derived from the inverse scattering series assume a constant background model (e.g. Shaw et al. 2003., H. Zhang and Weglein 2003). These sub-series demonstrate robust convergence when both the magnitude and duration of the difference between actual and reference are large. However, the next important practical issue concerns the rate of convergence. Early analysis and testing indicate that the rate of convergence will improve when the difference between actual and reference media is reduced.

Advances in multiple removal technology (See e.g. Weglein 1999) allow improved estimates of the subsurface velocity model. These two facts encourage the development of imaging and inversion sub-series that can accommodate a variable velocity model.

This paper represents the first effort in that direction. It provides the first two terms in the inverse series for a one parameter (variable velocity, constant density) acoustic medium. The reference medium $c_0(z)$ is assumed to be smooth, thereby allowing a WKBJ waveform to be appropriate. The separation of the second term in the $c_0(z)$ background inverse series into reflector imaging at depth is the near-term objective.

While the mathematical derivation of the second term is laborious and non-trivial the result is remarkably compact and can easily be shown to reduce to the earlier constant background case.

2 Motivation for using a closer approximation to the actual medium

With the discovery of the leading order imaging sub-series, which has the nice property of converging for any contrast, and the inversion sub-series, which is the only candidate for a direct inversion for multi-dimensional, corrugated, high-contrast Earth, a huge amount of additional computation is demanded. Only one term in the imaging or inversion sub-series may take CPU cycles equivalent to that of traditional migration. A typical 3D time migration may need several months to be executed on a super computer like SP2 or a PC cluster. That's why we need to accelerate the series summation by incorporating the roughly inadequate estimated velocity to form an adequate image.

How can we accelerate the series computation? One factor is the choice of the reference medium. The rate of convergence is much better if the reference medium is closer to the actual medium.

Constant background inversion assumes no a priori information is available. This nice property also has its negative side: the rate of convergence is relatively slow. After free-surface and internal multiples are removed, a more accurate velocity trend is available. We can incorporate this a priori information into the inverse scattering series to speed up the series calculation.

The use of $c_0(z)$ is motivated by the fact that, on a global scale, the structure of the Earth is dominated by the overwhelming influence of gravity. The structure of the earth can be much better approximated by a vertically varying but horizontally uniform model than a uniform velocity field filling the whole space. Our $c_0(z)$ reference model offers a much closer approximation to the actual earth.

3 Equations need to be solved:

Following Weglein et al. (1997), we will calculate the perturbation terms V_1, V_2, … using the inverse series:

$$
\begin{aligned}
D &= G_0 V_1 G_0 \\
0 &= G_0 V_2 G_0 + G_0 V_1 G_0 V_1 G_0 \\
0 &= G_0 V_3 G_0 + G_0 V_1 G_0 V_2 G_0 + G_0 V_2 G_0 V_1 G_0 + G_0 V_1 G_0 V_1 G_0 V_1 G_0 \\
&\bullet\bullet\bullet
\end{aligned}
\qquad (1)
$$

where D is the data obtained on the measurement surface, and all the expressions above are supposed to be evaluated on the measurement surface. G_0 is the Green's function of the reference medium, which cannot be easily obtained if the reference velocity is not constant or piecewise constant. In the next section, we explain the complexity of obtaining an analytical Green's function for an arbitrary medium.

4 WKBJ Green's function:

Let's consider the simplest case, when both the earth and the experiment are 1D. Then the Green's function satisfies the following inhomogeneous second-order differential equation (expressed in the frequency domain).

$$\frac{\partial^2 G(z,z_s,\omega)}{\partial z^2}+\left(\frac{\omega}{c(z)}\right)^2 G(z,z_s,\omega)=\delta(z-z_s) \tag{2}$$

where z_s is the coordinate of the source. In order to solve the equation above, we have to consider its corresponding homogeneous equation:

$$\frac{\partial^2 G(z,z_s,\omega)}{\partial z^2}+\left(\frac{\omega}{c(z)}\right)^2 G(z,z_s,\omega)=0 \tag{3}$$

It can be written in a more general form:

$$\frac{d^2 P(z)}{dz^2}+f(z)P(z)=0 \tag{4}$$

The seemingly very simple equation is of great importance in both mathematics and physics because any linear homogeneous second-order equation may be put in this form. Finding a precise solution to the problem of Green's function for variable background is mathematically equivalent to solving the problem above analytically. Since this problem had already been proved analytically unsolvable for arbitrary f(z), our precise Green's function is in general impossible to find.

In quantum mechanics, WKBJ methods had been developed by Wentzel, Kramers, Brillouin, Jeffreys to approximate the equation above. The WKBJ solutions to the homogeneous equation (4) (Methews and Walker, equation 1-90, page 28) are:

$$P(z)=\frac{1}{\sqrt[4]{f(z)}}\left\{c_1\exp\left(i\int^z\sqrt{f(x)}dx\right)+c_2\exp\left(-i\int^z\sqrt{f(x)}dx\right)\right\} \tag{5}$$

where c_1 and c_2 are arbitrary constants. Interested readers may refer to Yedlin's paper for constructing WKBJ Green's function in two dimensions.

Note that there are 2 linearly independent solutions to the homogeneous equation (3):

$$P(z,z_s,\omega)=\begin{cases} a_1\sqrt{\dfrac{c(z)}{\omega}}\exp\left(i\omega\displaystyle\int_{z_s}^{z}\frac{du}{c(u)}\right) \\ a_2\sqrt{\dfrac{c(z)}{\omega}}\exp\left(-i\omega\displaystyle\int_{z_s}^{z}\frac{du}{c(u)}\right)\end{cases} \tag{6}$$

where a_1 and a_2 are arbitrary constants. We are now in the position to construct the Green's function satisfying equation (2). It will consist of 2 parts, one for $z<z_s$, and one for $z>z_s$. The final result should be continuous at $z=z_s$, but the first derivative with respect to z will have a jump of magnitude 1, so the second derivative will have a δ-function behavior at $z=z_s$.

$$G(z,z_s,\omega)=\begin{cases} a_1\sqrt{\dfrac{c(z)}{\omega}}\exp\left(i\omega\displaystyle\int_{z_s}^{z}\frac{du}{c(u)}\right) & z>z_s \\ a_2\sqrt{\dfrac{c(z)}{\omega}}\exp\left(-i\omega\displaystyle\int_{z_s}^{z}\frac{du}{c(u)}\right) & z<z_s \end{cases} \tag{7}$$

Continuity at $z=z_s$ means that $a_1=a_2=a$. Now lets look at the left and right first derivative when $z=z_s$.

$$\begin{cases} \lim\limits_{z\to z_s^+}\dfrac{\partial G(z,z_s)}{\partial z}=\left.\dfrac{\partial G_1(z,z_s)}{\partial z}\right|_{z=z_s}=a\left\{i\sqrt{\dfrac{\omega}{c(z_s)}}+\dfrac{c'(z_s)}{2\sqrt{\omega c(z_s)}}\right\} \\ \lim\limits_{z\to z_s^-}\dfrac{\partial G(z,z_s)}{\partial z}=\left.\dfrac{\partial G_2(z,z_s)}{\partial z}\right|_{z=z_s}=a\left\{\dfrac{-i\omega}{\sqrt{c(z_s)}}+\dfrac{c'(z_s)}{2\sqrt{\omega c(z_s)}}\right\} \end{cases}$$

$$1=\lim_{z\to z_s^+}\frac{\partial G(z,z_s)}{\partial z}-\lim_{z\to z_s^-}\frac{\partial G(z,z_s)}{\partial z}=i2a\sqrt{\frac{\omega}{c(z_s)}}$$

$$\Rightarrow \qquad a=\frac{1}{2i}\sqrt{\frac{c(z_s)}{\omega}}$$

So we have our causal WKBJ Green's function:

$$G(z,z_s,\omega)=\frac{\sqrt{c(z)c(z_s)}}{2i\omega}\exp\left(i\omega\left|\int_{z_s}^{z}\frac{du}{c(u)}\right|\right) \tag{8}$$

5 Solving the problems with WKBJ Green's function:

5.1 Solving for $\alpha_1(z)$

The first problem stated in equation (1) is:

$$D(z_g,z_s,\omega)=\int_{-\infty}^{\infty}dz'\,G_0(z_g,z',\omega)V_1(z')G_0(z',z_s,\omega) \tag{9}$$

Without losing any generality, we assume both the source and the receiver located at 0 depth, that is: $z_g=z_s=0$. As in the case of constant background, let's define:

$$\alpha_1(z) = \left(\frac{\omega}{c(z)}\right)^{-2} V_1(z). \tag{10}$$

We can also define a function similar to travel time in the reference medium, but having a negative value when traveling upwards.

$$\tau(z_1, z_2) = \int_{z_1}^{z_2} \frac{du}{c(u)} \tag{11}$$

After applying a Fourier transform we change equation (9) into:

$$\int_{-\infty}^{\infty} d\omega D(0,0,\omega) e^{-i\omega 2\tau(0,z)} = \int_{-\infty}^{\infty} d\omega \left\{ \int_{-\infty}^{\infty} dz' G_0(0,z',\omega) V_1(z') G_0(z',0,\omega) \right\} e^{-i\omega 2\tau(0,z)} \tag{12}$$

Solving the equation above (see Appendix A), we obtain:

$$-\frac{c(0)\pi}{4}\alpha_1(z) = \int_{-\infty}^{\infty} d\omega D(0,0,\omega) e^{-i\omega 2\tau(0,z)} \tag{13}$$

We can also use other incident wave-fields, the one commonly used in 1D constant background scattering series can be obtained by multiplying G_0 with 2ik. It's better than G_0 itself in presenting data in the time domain because its time-domain representation is a spike, but that of G_0 is a Heaviside function. A more realistic wavelet in seismic exploration would be G_0 multiplied by 2ik. So the data in the frequency domain will also be multiplied by 2ik =2iω/c(0), let's denote this data after changing wavelet by $D_w(z_g, z_s, \omega)$. We have:

$$D_W(0,0,\omega) = D(0,0,\omega)\frac{2i\omega}{c(0)}$$

$$D(0,0,\omega) = \frac{D_W(0,0,\omega)}{i2\omega / c(0)}$$

$$\alpha_1(z) = -\frac{4}{c(0)\pi} \int_{-\infty}^{\infty} d\omega \frac{D_W(0,0,\omega)}{i2\omega / c(0)} e^{-i\omega 2\tau(0,z)} \xrightarrow{t=2\tau(0,z)} -\frac{4}{\pi} \int_{-\infty}^{\infty} d\omega \frac{D_W(0,0,\omega)}{i2\omega} e^{-i\omega t}$$

$$\frac{d\alpha_1(z)}{dt} = \frac{2}{\pi} \int_{-\infty}^{\infty} d\omega D_W(0,0,\omega) e^{-i\omega t} = \frac{2}{\pi} \int_{-\infty}^{\infty} d\omega D_W(0,0,\omega) e^{-i\omega t}$$

$$\frac{d\alpha_1(z)}{dt} = \frac{2}{\pi} \int_{-\infty}^{\infty} d\omega D_W(0,0,\omega) e^{-i\omega t} = 4\widetilde{D}_W(0,0,t)$$

$$\alpha_1(z) = 4 \int_{-\infty}^{2\tau(0,z)} \widetilde{D}_W(0,0,t) dt \tag{14}$$

where $\widetilde{D}_W(0,0,t)$ is the seismic trace in the time domain. Just as in the case of constant background, α_1 is a trace integral in the time domain. The α_1 formula in constant background can be expressed as:

$$\alpha_1(z) = \frac{8}{c_0} \int_{-\infty}^{2z/c_0} \widetilde{D}(0,0,t)dt$$

It positions the events in the time domain uniformly into space according to the reference velocity c_0. Compared with the corresponding α_1 formula, equation (14) has the flexibility to position the events non-uniformly in time domain according to a variable velocity trend, an additional power looks like migration.

5.2 Solving for $\alpha_2(z)$

The second problem stated in equation (1) is:

$$G_0 V_2 G_0 = -G_0 V_1 G_0 V_1 G_0$$

Applying the same Fourier transform as before, we have:

$$\int_{-\infty}^{\infty} G_0 V_2 G_0 e^{-i\omega 2\tau(0,z)} d\omega = \int_{-\infty}^{\infty} - G_0 V_1 G_0 V_1 G_0 e^{-i\omega 2\tau(0,z)} d\omega \tag{15}$$

Solving for the equation above (see Appendix B), we have:

$$-\frac{c(0)\pi}{4}\alpha_2(z) = \frac{c(0)\pi}{8}\left\{\alpha_1^2(z) + c(z)\alpha'_1(z)\int_{-\infty}^{z} dz' \frac{\alpha_1(z')}{c(z')}\right\}$$

So we can express $\alpha_2(z)$ (in terms of $\alpha_1(z)$) as:

$$\alpha_2(z) = -\frac{1}{2}\left\{\alpha_1^2(z) + c(z)\alpha'_1(z)\left\{\int_{-\infty}^{z} dz' \frac{\alpha_1(z')}{c(z')}\right\}\right\} \tag{16}$$

In a special case of constant background, $c'(z) \equiv 0$, we have:

$$\alpha_2(z) = -\frac{1}{2}\left\{\alpha_1^2(z) + \alpha'_1(z)\left\{\int_{-\infty}^{z} dz' \alpha_1(z')\right\}\right\} \tag{17}$$

Equation (17) agrees with the $\alpha_2(z)$ term in the constant background inversion. If $c'(z) \neq 0$, there will be additional terms in equation (16). We will study these terms further and identify their use in imaging and inversion.

6 Future work

Future work will include: (1) Calculating terms beyond $\alpha_2(z)$. (2) Identifying the specific sub-series that perform different inversion tasks. (3) Numerically testing the convergence properties of these sub-series. (4) Working out the corresponding terms in multi-dimensions, which are currently approximately solved only for the first (linear) term.

7 Conclusion

Two terms in the inversion scattering series with variable background have been calculated with a WKBJ Green's function in 1D. The inverse equations can be precisely solved using Fourier transform. The first term introduces more flexibility to position the events in the time domain non-uniformly by a varying velocity trend. More work, including numerical tests, is needed to study newly discovered terms.

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Appendix A --- deriving $\alpha_1(z)$

Using the WKBJ Green's function (7), we have:

$$\int_{-\infty}^{\infty} dz' G_0(0,z',\omega)V_1(z')G_0(z',0,\omega)$$
$$= \int_{-\infty}^{\infty} dz' \frac{1}{2i}\sqrt{\frac{c(0)}{\omega}\frac{c(z')}{\omega}} \exp\left(i\omega\left|\int_{z'}^{0}\frac{dz''}{c(z'')}\right|\right)\left(\frac{\omega}{c(z')}\right)^2 \alpha_1(z')\frac{1}{2i}\sqrt{\frac{c(0)}{\omega}\frac{c(z')}{\omega}} \exp\left(i\omega\left|\int_{z'}^{0}\frac{dz''}{c(z'')}\right|\right)$$

We assume that no perturbation occurs for $z<0$, that is: $z<0 \Rightarrow \alpha_1(z)=0$, we can simplify the expression above further:

$$= -\frac{1}{4}\int_{-\infty}^{\infty} dz' \exp\left(i2\omega\int_{0}^{z'}\frac{dz''}{c(z'')}\right)\frac{c(0)\alpha_1(z')}{c(z')} = -\frac{c(0)}{4}\int_{-\infty}^{\infty} dz' \exp(i2\omega\tau(0,z'))\frac{\alpha_1(z')}{c(z')}$$

So the right-hand-side of (10a) is:

$$\int_{-\infty}^{\infty} d\omega\left\{\int_{-\infty}^{\infty} dz' G_0(0,z',\omega)V_1(z')G_0(z',0,\omega)\right\}e^{-i\omega 2\tau(0,z)}$$
$$= \int_{-\infty}^{\infty} d\omega\left\{-\frac{c(0)}{4}\int_{-\infty}^{\infty} dz' e^{i2\omega\tau(0,z')}\frac{\alpha_1(z')}{c(z')}\right\}e^{-i\omega 2\tau(0,z)}$$

By switching the order of integration, we have:

$$= -\frac{c(0)}{4}\int_{-\infty}^{\infty} dz' \frac{\alpha_1(z')}{c(z')}\int_{-\infty}^{\infty} d\omega e^{i\omega\{2\tau(0,z')-2\tau(0,z)\}}$$

$$= -\frac{c(0)}{4}(2\pi)\int_{-\infty}^{\infty} dz' \frac{\alpha_1(z')}{c(z')}\delta(2\tau(0,z') - 2\tau(0,z))$$
$$= -\frac{c(0)\pi}{2}\int_{-\infty}^{\infty} dz' \frac{\alpha_1(z')}{c(z')}\delta(2\tau(0,z) - 2\tau(0,z'))$$

Let's apply the formula for integration with δ-function (A-1) in Appendix 3 (here we take $\psi(z) = \frac{\alpha_1(z)}{c(z)}$). We have:

$$-\frac{c(0)\pi}{2}\int_{-\infty}^{\infty} dz' \frac{\alpha_1(z')}{c(z')}\delta(2\tau(0,z) - 2\tau(0,z'))$$
$$= -\frac{c(0)\pi}{2}\frac{\alpha_1(z)}{c(z)}\frac{c(z)}{2} = -\frac{c(0)\pi}{4}\alpha_1(z)$$

Appendix B --- deriving $\alpha_2(z)$

The left-hand-side of (13) can be similarly calculated as in Appendix A. The only difference is changing 1 to 2, all other derivations are the same. Here we give the final result:

$$-\frac{c(0)\pi}{4}\alpha_2(z).$$

But the right-hand-side is more tedious, the derivation procedure is:

$$G_0V_1G_0V_1G_0$$
$$= \int_{-\infty}^{\infty} dz' G_0(z_g, z', \omega)\left(\frac{\omega}{c(z')}\right)^2 \alpha_1(z')\int_{-\infty}^{\infty} dz'' G_0(z', z'', \omega)\left(\frac{\omega}{c(z'')}\right)^2 \alpha_1(z'')G_0(z'', z_s, \omega)$$
$$= \int_{-\infty}^{\infty} dz' \frac{1}{2i}\sqrt{\frac{c(0)c(z')}{\omega^2}}e^{i\omega|\tau(0,z')|}\left(\frac{\omega}{c(z')}\right)^2 \alpha_1(z')\int_{-\infty}^{\infty} dz'' \frac{1}{2i}\sqrt{\frac{c(z')c(z'')}{\omega^2}}e^{i\omega|\tau(z',z'')|}\left(\frac{\omega}{c(z'')}\right)^2 \alpha_1(z'')\frac{1}{2i}\sqrt{\frac{c(0)c(z'')}{\omega^2}}e^{i\omega|\tau(z'',0)|}$$
$$= -\frac{c(0)}{8i}\omega\int_{-\infty}^{\infty} dz' e^{i\omega|\tau(0,z')|}\frac{\alpha_1(z')}{c(z')}\int_{-\infty}^{\infty} dz'' e^{i\omega|\tau(z',z'')|}\frac{\alpha_1(z'')}{c(z'')}e^{i\omega|\tau(0,z'')|}$$

We assume that no perturbation occurs for $z<0$, that is: $z < 0 \Rightarrow \alpha_1(z) = 0$, we can simplify the expression above further:

$$= -\frac{c(0)}{8i}\omega\int_{-\infty}^{\infty} dz' e^{i\omega\tau(0,z')}\frac{\alpha_1(z')}{c(z')}\int_{-\infty}^{\infty} dz'' e^{i\omega|\tau(z',z'')|}\frac{\alpha_1(z'')}{c(z'')}e^{i\omega\tau(0,z'')}$$

By defining: $\psi_1(z) = \frac{\alpha_1(z)}{c(z)}$, we have:

$$= -\frac{c(0)}{8i}\omega \int_{-\infty}^{\infty} dz' e^{i\omega\tau(0,z')}\psi_1(z')\int_{-\infty}^{\infty} dz'' \{H(z''-z')e^{i\omega\tau(z',z'')} + H(z'-z'')e^{i\omega\tau(z'',z')}\}\psi_1(z'')e^{i\omega\tau(0,z'')}$$
$$= \frac{c(0)}{8}(i\omega)\int_{-\infty}^{\infty} dz' e^{i\omega\tau(0,z')}\psi_1(z')\int_{-\infty}^{\infty} dz'' \{H(z''-z')e^{i\omega\tau(z',z'')} + H(z'-z'')e^{i\omega\tau(z'',z')}\}\psi_1(z'')e^{i\omega\tau(0,z'')}$$

Now we can calculate the Fourier transform of the right-hand-side:

$$\int_{-\infty}^{\infty}(-G_0V_1G_0V_1G_0)e^{-i2\omega\tau(0,z)}d\omega = \frac{c(0)}{8}\{I_1 + I_2\}$$

where I_1 is calculated as:

$$I_1 = -\int_{-\infty}^{\infty}\left\{i\omega\int_{-\infty}^{\infty} dz'\psi_1(z')\int_{-\infty}^{\infty} dz''\{H(z''-z')\}\psi_1(z'')e^{i\omega 2\tau(0,z'')}\right\}e^{-i2\omega\tau(0,z)}d\omega$$
$$= \int_{-\infty}^{\infty} dz'\psi_1(z')\int_{-\infty}^{\infty} dz'' H(z''-z')\psi_1(z'')\int_{-\infty}^{\infty} d\omega(-i\omega)e^{-i\omega\{2\tau(0,z)-2\tau(0,z'')\}}$$
$$= (2\pi)\int_{-\infty}^{\infty} dz'\psi_1(z')\int_{-\infty}^{\infty} dz'' H(z''-z')\psi_1(z'')\delta'(2\tau(0,z)-2\tau(0,z''))$$

Let's apply the formula for integration with δ-function (A-2) in Appendix C (here we take $\psi(z'') = \psi_1(z'')H(z''-z') = \frac{\alpha_1(z'')}{c(z'')}H(z'-z'')$). The expression above can be simplified further as:

$$= \frac{c^2(z)\pi}{2}\int_{-\infty}^{\infty} dz'\psi_1(z')\{\delta(z-z')\psi_1(z) + H(z-z')\psi'_1(z)\}$$
$$+ \frac{c(z)c'(z)\pi}{2}\int_{-\infty}^{\infty} dz'\psi_1(z')H(z-z')\psi_1(z)$$
$$= \frac{c^2(z)\pi}{2}\left\{\psi_1^2(z) + \psi'_1(z)\int_{-\infty}^{z} dz'\psi_1(z')\right\} + \frac{c(z)c'(z)\pi}{2}\psi_1(z)\int_{-\infty}^{z} dz'\psi_1(z')$$
$$= \frac{\pi}{2}\{c^2(z)\psi_1^2(z) + \} + \frac{\pi}{2}c(z)[c(z)\psi'_1(z) + c'(z)\psi_1(z)]\int_{-\infty}^{z} dz'\psi_1(z')$$
$$= \frac{\pi}{2}\{c^2(z)\psi_1^2(z)\} + \frac{\pi}{2}c(z)[c(z)\psi_1(z)]'\int_{-\infty}^{z} dz'\psi_1(z')$$
$$= \frac{\pi}{2}\{\alpha_1^2(z)\} + \frac{\pi}{2}c(z)[\alpha_1(z)]'\int_{-\infty}^{z} dz'\psi_1(z')$$

$$= \frac{\pi}{2}\left\{\alpha_1^2(z)\right\} + \frac{\pi}{2} c(z)\alpha'_1(z)\int_{-\infty}^{z} dz' \frac{\alpha'_1(z')}{c(z')}$$

$$= \frac{\pi}{2}\left\{\alpha_1^2(z) + c(z)\alpha'_1(z)\int_{-\infty}^{z} dz' \frac{\alpha'_1(z')}{c(z')}\right\}$$

Similarly, I_2 can be calculated as:

$$I_2 = -\int_{-\infty}^{\infty}\left\{ i\omega \int_{-\infty}^{\infty} dz' \psi_1(z') \int_{-\infty}^{\infty} dz'' \{H(z'-z'')\}\psi_1(z'') e^{i\omega 2\tau(0,z')}\right\} e^{-i2\omega\tau(0,z)} d\omega$$

$$= \int_{-\infty}^{\infty} dz' \psi_1(z') \int_{-\infty}^{\infty} dz'' H(z'-z'')\psi_1(z'') \int_{-\infty}^{\infty} d\omega (-i\omega) e^{-i\omega\{2\tau(0,z)-2\tau(0,z')\}}$$

$$= (2\pi)\int_{-\infty}^{\infty} dz' \psi_1(z') \int_{-\infty}^{\infty} dz'' H(z'-z'')\psi_1(z'')\delta'(2\tau(0,z) - 2\tau(0,z'))$$

$$= \int_{-\infty}^{\infty} dz'' \psi_1(z'') \int_{-\infty}^{\infty} dz' \psi_1(z') H(z'-z'')\delta'(2\tau(0,z) - 2\tau(0,z'))$$

Let's apply the formula for integration with δ-function (A-2) in Appendix C (here we take $\psi(z') = \psi_1(z')H(z'-z'') = \frac{\alpha_1(z')}{c(z')} H(z''-z')$). We have:

$$= \frac{c^2(z)\pi}{2}\int_{-\infty}^{\infty} dz'' \psi_1(z'')\{\delta(z - z'')\psi_1(z) + H(z - z'')\psi'_1(z)\}$$

$$+ \quad \frac{c(z)c'(z)\pi}{2}\psi_1(z)\int_{-\infty}^{\infty} dz'' \psi_1(z'') H(z - z'')$$

$$= \frac{c^2(z)\pi}{2}\left\{\psi_1^2(z) + \psi'_1(z)\int_{-\infty}^{z} dz'' \psi_1(z'')\right\} + \frac{c(z)c'(z)\pi}{2}\psi'_1(z)\int_{-\infty}^{z} dz'' \psi_1(z'')$$

$$= \frac{\pi}{2}\left\{c^2(z)\psi_1^2(z)\right\} + \frac{c(z)\pi}{2}\left\{(c(z)\psi'_1(z) + c'(z)\psi_1(z))\int_{-\infty}^{z} dz'' \psi_1(z'')\right\}$$

$$= \frac{\pi}{2}\left\{c^2(z)\psi_1^2(z)\right\} + \frac{c(z)\pi}{2}\left\{(c(z)\psi_1(z))' \int_{-\infty}^{z} dz'' \psi_1(z'')\right\}$$

$$= \frac{\pi}{2}\left\{c^2(z)\psi_1^2(z)\right\} + \frac{c(z)\pi}{2}\left\{\alpha'_1(z)\int_{-\infty}^{z} dz' \frac{\alpha_1(z')}{c(z')}\right\}$$

$$= \frac{\pi}{2}\left\{\alpha_1^2(z) + \alpha'_1(z)c(z)\int_{-\infty}^{z} dz' \frac{\alpha_1(z')}{c(z')}\right\}$$

In summary, we have:

$$\int_{-\infty}^{\infty}(-G_0V_1G_0V_1G_0)e^{-i2\omega\tau(0,z)}d\omega=\frac{c(0)}{8}\{I_1+I_2\}$$

$$=\frac{c(0)\pi}{8}\left\{\alpha_1^2(z)+\alpha'_1(z)c(z)\int_{-\infty}^{z}dz'\frac{\alpha_1(z')}{c(z')}\right\}$$

Appendix C --- Integral with δ-function and its first derivative

In this paper, the function inside δ-function and it's first derivative will always be the 2-way travel time between 0 and z:

$$f(z)=2\int_0^z\frac{dz'}{c(z')}=2\tau(0,z)$$

where z is depth, c(u) is a function of reference velocity. Let's consider it's relation with it's inverse (denoted here by g):

$u=f(z)$ and $z=g(u)=f^{-1}(u)$

If $u_0=f(z_0)$. We have: $g'(u_0)=\dfrac{1}{f'(z_0)}$

Now it's time to calculate the following integral:

$$\int_{-\infty}^{\infty}dz\{\psi(z)\delta(f(z_0)-f(z))\}$$

where ψ(z) is an arbitrary function.

After changing integration variable: $u=f(z)$ and $u_0=f(z_0)$

We have $dz=g'(u)du$,

$$\int_{-\infty}^{\infty}dz\{\psi(z)\delta(f(z_0)-f(z))\}=\int_{-\infty}^{\infty}du\{\psi(g(u))\delta(u_0-u)\}g'(u)$$

$$=\int_{-\infty}^{\infty}du\{\psi(g(u))g'(u)\delta(u_0-u)\}$$

$$=\psi(g(u_0))g'(u_0)=\psi(f^{-1}(u_0))g'(u_0)$$

$$=\psi(z_0)g'(u_0)=\psi(z_0)\frac{1}{f'(z_0)}=\frac{\psi(z_0)}{f'(z_0)}=\frac{\psi(z_0)c(z_0)}{2}\qquad\text{(A-1)}$$

Now it's time to calculate the integral with the first derivative of δ-function:

$$\int_{-\infty}^{\infty} dx\{\psi(z)\delta'(f(z_0)-f(z))\}$$

where ψ(z) is an arbitrary function.

After changing integration variable: $u=f(z)$ and $u_0=f(z_0)$

We have $dz=g'(u)du$,

$$\int_{-\infty}^{\infty} dx\{\psi(z)\delta'(f(z_0)-f(z))\}=\int_{-\infty}^{\infty} du\{\psi(g(u))g'(u)\delta'(u_0-u)\}$$

Denote: $\varphi(u)=\psi(g(u))g'(u)$

We have:

$$\int_{-\infty}^{\infty} du\{\psi(g(u))g'(u)\delta'(u_0-u)\}=\int_{-\infty}^{\infty} du\{\varphi(u)\delta'(u_0-u)\}$$

$$=-\int_{-\infty}^{\infty}\varphi(u)d\delta'(u_0-u)=\left[-\varphi(u)\delta(u_0-u)\right]_{x=-\infty}^{x=\infty}+\int_{-\infty}^{\infty}\delta(u_0-u)d\varphi(u)$$

$$=\int_{-\infty}^{\infty}\delta(u_0-u)\varphi'(u)du=\varphi'(u_0)$$

$$=\frac{d}{du}\{\psi(g(u))g'(u)\}\Big|_{u=u_0}$$

$$=\psi'(g(u))[g'(u)]^2+\psi(g(u))g''(u)\Big|_{u=u_0}$$

$$=\psi'(g(u_0))[g'(u_0)]^2+\psi(g(u_0))g''(u_0)$$

$$=\psi'(f^{-1}(u_0))[g'(u_0)]^2+\psi(f^{-1}(u_0))g''(u_0)$$

$$=\frac{\psi'(x_0)}{[f'(x_0)]^2}+\psi(x_0)g''(u_0)$$

In the expression above, $g''(u_0)$ can be calculated by:

$$g''(u)=\frac{d}{du}\{g'(u)\}$$

Because: $g=f^{-1}$, we have:

$$g''(u)=\frac{d}{du}\left\{\frac{1}{f'(z)}\right\}=\frac{d}{dz}\left\{\frac{1}{f'(z)}\right\}/\frac{du}{dz}=\frac{d}{dz}\left\{\frac{1}{f'(z)}\right\}\frac{1}{f'(z)}=\frac{-f''(z)}{[f'(z)]^3}$$

So we have:

$$\int_{-\infty}^{\infty} du\{\psi(g(u))g'(u)\delta'(u_0-u)\}=\frac{\psi'(x_0)}{[f'(x_0)]^2}+\psi(x_0)g''(u_0)$$

$$= \frac{\psi'(z_0)}{[f'(z_0)]^2} - \psi(z_0)\frac{f''(z_0)}{[f'(z_0)]^3} = \frac{\psi'(z_0)c^2(z_0)}{4} + \frac{\psi(z_0)c^3(z_0)}{8}\frac{2c'(z_0)}{c^2(z_0)}$$

$$= \frac{c(z_0)}{4}\{\psi'(z_0)c(z_0) + \psi(z_0)c'(z_0)\} \tag{A-2}$$