Short note: G_0^{DD} and G_0^D integral equations relationships; The triangle relation is intact

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Abstract

Weglein and Secrest (1990) established a relationship between the total pressure wavefield measured on the cable, the vertical derivative of that wavefield and the source signature. Amundsen et al (1995) and Corrigan et al (1991) use the triangle relationship to solve for $\frac{dP}{dn}$ from $A(\omega)$ and P along the cable. Tan (1992, 1999) and Osen et al. (1998) establish a relationship between the pressure wavefield along the cable, a single pressure measurement, between the free-surface and the cable, and the wavelet. We demonstrate how these two formulations relate to each other, and in that process establish that, in fact, they correspond to the same relationship between P and P', along the cable, and the wavelet, $A(\omega)$.

1 Definition

We present this analysis for a 1D version of these extinction theorem applications, for estimating the wavelet, since we understand that a Fourier transform over the lateral spatial variables in 2D and 3D leads to this precise 1D form (where $k_z = \sqrt{(\omega/c_0)^2 - k_x^2 - k_y^2}$ would substitute for $k = \omega/c_0$ in this paper) and z and z' would be depth variables.

We begin by defining G_0, G_0^D and G_0^{DD} as the causal whole space Green's function, the Green's function with Dirichlet boundary conditions on the free-surface, and the "double Dirichlet" Green's function vanishing on the free-surface and cable, respectively.

$G_0(z, z', \omega)$ satisfies the differential equation

$$\left(\frac{d^2}{dz^2} + k^2\right) G_0(z, z', \omega) = \delta(z - z') \tag{1}$$

for $0 \leq z, z' \leq a$ where $z = a$ is the measurement surface and $z = 0$ is the free surface. G_0^{DD} is the solution to equation (1) that vanishes at $z = 0$ and $z = a$. The general solution to (1) for any boundary conditions is

$$Ae^{ikz} + Be^{-ikz} + \frac{e^{ik|z-z'|}}{2ik} = G_0(z, z', \omega) \tag{2}$$

for $0 \le z, z' \le a$ and imposing

$$G_0(0, z', \omega) = 0 \quad (3)$$
$$G_0(a, z', \omega) = 0 \quad (4)$$

we find $G_0^{DD}(z, z', \omega)$ (the double D superscript denotes Dirichlet boundary conditions on two surfaces, $z = 0$ and $z = a$) to be

$$\begin{aligned} G_0^{DD}(z, z', \omega) &= \frac{1}{2ik}\left[\frac{e^{ik(z'-a)} - e^{-ik(z'-a)}}{e^{ika} - e^{-ika}}\right] e^{ikz} \\ &\quad -\frac{1}{2ik}\left[\frac{e^{ika}\left(e^{ikz'} - e^{-ikz'}\right)}{e^{ika} - e^{-ika}}\right] e^{-ikz} + \frac{e^{ik|z-z'|}}{2ik}. \end{aligned} \quad (5)$$

Equation (5) can be combined

$$G_0^{DD} = \frac{1}{2ik}\frac{e^{ik(z-z')}}{1 - e^{2ika}}\left(1 - e^{2ikz'}\right)\left(1 - e^{2ik(a-z)}\right). \quad (6)$$

The last expression for G_0^{DD} is only for field point, z, below the source point z'. The Weglein-Secrest (1990) G_0^D result is

$$\begin{aligned} P(z, z_s, \omega) = &A(\omega) G_0^D(z, z_s, \omega) \\ &+ P(a, z_s, \omega)\left[\frac{d}{dz'} G_0^D(z, z', \omega)\right]_{z'=a} \\ &- P'(a, z_s, \omega)\, G_0^D(z, a, \omega) \end{aligned} \quad (7)$$

where $G_0^D(z, z', \omega)$ vanishes only at $z = 0$.

2 The Osen et al. and H. Tan result using G_0^{DD}

For $0 < z < a$

$$P(z, z_s, \omega) = A(\omega) G_0^{DD}(z, z_s, \omega) + P(a, z_s, \omega)\left[\frac{d}{dz'} G_0^{DD}(z, z', \omega)\right]_{z'=a} \quad (8)$$

If we take the limit as $z \to a$ in (8) we find $P(a, z_s, \omega) = P(a, z_s, \omega)$ since

$$\lim_{z \to a}\left[\frac{d}{dz'} G_0^{DD}(z, z', \omega)\right]_{z'=a} = 1.$$

For the multi D case, it means that as the point above the cable approaches the cable, the G_0^{DD} vanishes and the $\frac{d}{dn}G_0^{DD}$ becomes δ-like. Hence, the field above the cable will only depend on one point that approaches the cable. This is a reassuring result!

If we differentiate (8) by z

$$P'(z, z_s, \omega) = A(\omega)\frac{dG_0^{DD}(z, z_s, \omega)}{dz} + P(a, z_s, \omega)\frac{d}{dz}\left[\frac{d}{dz'}G_0^{DD}(z, z', \omega)\right]_{z'=a} \tag{9}$$

and then take the limit as $z \to a$

$$P'(a, z_s, \omega) = A(\omega)\left[\frac{dG_0^{DD}(z, z_s, \omega)}{dz}\right]_{z=a} + P(a, z_s, \omega)\left[\frac{d}{dz}\left[\frac{d}{dz'}G_0^{DD}(z, z', \omega)\right]_{z'=a}\right]_{z=a} \tag{10}$$

Equation (10) is a relationship between $P(a, z_s, \omega)$, $P'(a, z_s, \omega)$ and $A(\omega)$. From (7), we can evaluate as $z \to a$

$$\begin{aligned} P(a, z_s, \omega) &= A(\omega)G_0^D(a, z_s, \omega) + P(a, z_s, \omega)\left[\frac{d}{dz'}G_0^D(a, z', \omega)\right]_{z'=a} \\ &\quad - P'(a, z_s, \omega)\, G_0^D(a, a, \omega) \end{aligned} \tag{11}$$

Hence (10) and (11) are both relations between $P(a, z_s, \omega)$, $P'(a, z_s, \omega)$ and $A(\omega)$. Are equations (10) and (11) independent relationships between P, P' and $A(\omega)$? E.g., if true, we could derive an exact relationship for $A(\omega)$ directly in terms of P. Or P' from P exactly.

It is useful in comparing (10) and (11) to notice that

$$\frac{G_0^D(a, z_s, \omega)}{G_0^D(a, a, \omega)} = \left(\frac{d}{dz}G_0^{DD}(z, z_s, \omega)\right)_{z=a}$$

where

$$G_0^D(z, z', \omega) = \frac{e^{ik(z-z')}}{2ik}\left(1 - e^{2ikz'}\right),$$

again for $z > z'$; it shows that these two integral expressions (11) and (10) are identical.

Hence, the triangle holds!

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